ANALYTICAL SYNTHESIS OF LEAST CURVATURE $2D$ PATHS FOR UNDERWATER APPLICATIONS

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Calculating smooth trajectories of bounded curvature has received a very wide attention in the robotic literature of the last 40 years. Most results focus on the study of wheeled non-holonomic vehicles and are concerned with $2D$ path generation algorithms. The problem of smooth and least curvature $3D$ path planning is addressed with a variational approach and the general $3D$ Euler-Poisson equation is derived. The $2D$ solution is calculated as the plane projection of the general $3D$ solution and some special $2D$ cases are analyzed. It is shown that if some special conditions are satisfied along the $2D$ path the optimal solution is approximated by the well known Cornu spiral; moreover in these same conditions the optimal solution (and the Cornu spiral) are approximated by the more familiar cubic polynomial. Applications to rigid body underwater dynamics are discussed.

1 Introduction

The problem of finding a smooth and minimum curvature trajectory between two given configurations (a *configuration* is a point and direction in the plane) has received a wide attention in the robotic literature, specially regarding the steering of non-holonomic mobile robots, but has also been discussed within other fields such as civil engineering or computer graphics. Minimizing the path's curvature is a very interesting planning criterion also for rigid bodies moving in a fluid environment, as objects grasped by underwater manipulators or underwater vehicles. Pressure drag (sometimes called form drag) and added mass hydrodynamic loads are proportional to the surface of attach of the moving body, thus if the velocity vector is kept parallel to the major body surface, minimizing the curvature will minimize fluid forces applied on the body during curves. From the pioneering work of Dubins [1] who calculated the shortest path of bounded curvature among two configurations, many other authors focused their attention on the generation of bounded curvature $2D$ paths. In synthesis Dubins' results state that the shortest $2D$ path of bounded curvature between to fixed configurations may be traced joining straight lines and circular arcs of curvature smaller or equal to the maximum allowed. More recently these results have been discussed and elegantly refined by J.D.Boissonnat et al. [2], X.N.Bui et al. [3] and A.M.Shkel et al. [4]. Kanayama et al. [5] suggest the use of paths generated joining cubic spirals and arc of circles to minimize two cost functions related to curvature and jerk energies while A.M.Hussein et al. [6] generate smooth paths optimizing the integral of the square acceleration instead of curvature. One of the cost functions used by Kanayama et al. [5], and that is at the center of the present paper, is the integral over the path's lengths of its square curvature. The minimization of such quantity with fixed boundary configurations is a problem with an interest of its own as such cost function can be physically interpreted as proportional to the elastic energy of the curve. Due to this fact the sought plane path is sometimes called the *least energy curve* in literature. Indeed this interpretation makes the problem appealing also to researchers of other fields as A.M.Bruckstein et al. [7], B.K.P.Horn [8] and M.Kallay [12] who addressed a very similar problem to the one here discussed within a different framework and formulation. It will be shown that Horn's [8] and Kallay's [12] $2D$ results can be viewed as the projection on a plane of a more general $3D$ Euler-Poisson equation. In Section 2 the problem is formally stated and underwater applications are explicitly discussed. In Section 3 the general $3D$ solution is derived through variational calculus and some $2D$ special cases are discussed. In Section 4 the main $2D$ solution properties are discussed while in Section 5 some $2D$ examples and results are shown. Section 6 focuses on the concluding remarks.

2 Problem statement

Given a rigid body underwater moving in the horizontal plane $z = constant$ its dynamic equations with respect to the local reference frame with origin in the center of mass can be theoretically deduced under some standard hypothesis [9]: the fluid is ideal, i.e., of constant and uniform density, irrotational and inviscid, unbounded and of infinite extent except for the rigid body itself. The kinematics variables surge u, sway v and yaw $r = \dot{\theta}$ are velocities respect to the fluid. Neglecting time varying currents the hydrodynamic load in deep water ($\geq 5m$) where wave effects are virtually absent is due to drag, lift and added mass forces. Drag is anti-parallel to the velocity and drag coefficients are proportional to the surface of attach. Lift is normal to the velocity direction, proportional to its value and to the angle of attach provided it is small enough ($\leq 12^o$ as an order of magnitude, stall occurs for higher values). Added mass forces are proportional to accelerations $\frac{du}{dt}, \frac{dv}{dt}, \frac{dr}{dt}$ through added mass coefficients which depend on the body's shape. To avoid large sway drag forces and surge added mass forces that cause major hydrodynamic load on an elongated body the lateral sway velocity v and the linear acceleration $\frac{du}{dt}$ should be kept null. Note that the constraint on null sway makes the present problem very similar to the well known non-holonomic car-like path planning problem. Yaw velocity r and acceleration $\frac{dr}{dt}$ should be minimized as the large lateral surface produces strong moments along the z axis. Lift forces can be controlled through the value of surge velocity u. Thus assuming that $v = 0$ and that surge u velocity is kept constant and small to avoid added mass stresses and limit lift effects, the major dissipative force acting on the body will be caused by drag rotation moment in the z direction that is linear in r. The energy associated with such drag moment is proportional to $\int r d\theta = \int r \frac{d\theta}{ds} ds = \int rk ds = u \int k^2 ds$ where s is the curvilinear coordinate, $k = r/u$ the paths curvature and u the constant surge velocity. Indicating from here on with a dot the derivative respect to s and with $(x(s), y(s))$ the unknown path of length L, unit tangent vector $\mathbf{T}(s) = (\dot{x}(s), \dot{y}(s))$ and curvature $k(s) = \left\| \dot{\mathbf{T}}(s) \right\| = \sqrt{\ddot{x}^2(s) + \ddot{y}^2(s)}$ the following cost function to be minimized with relative boundary conditions is defined

$$\varepsilon \equiv \int_0^L k^2 ds \tag{1}$$

$$\begin{cases} x(0) = x_0 \ ; \ y(0) = y_0 \\ x(L) = x_f \ ; \ y(L) = y_f \\ \frac{\dot{y}(0)}{\dot{x}(0)} = \tan(\theta_0) \ ; \frac{\dot{y}(L)}{\dot{x}(L)} = \tan(\theta_f) \end{cases} \tag{2}$$

Where θ_0 and θ_f are the initial and final angles between the curve and the x-axis. Notice that L is not fixed and

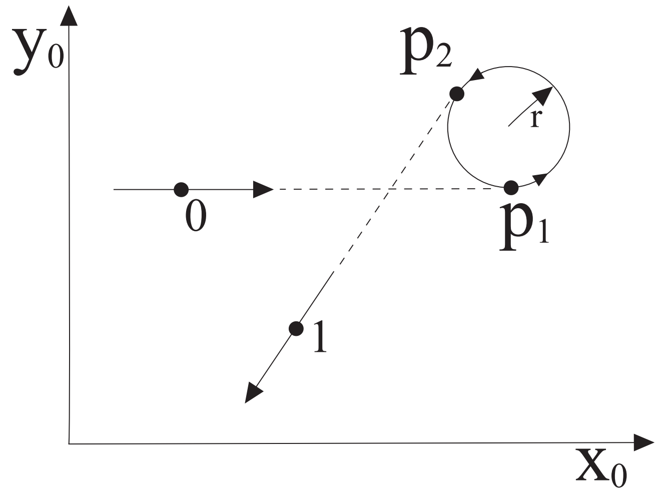

Figure 1: Infinite length solutions

if $L \to \infty$ it is always possible to find a path for which $\varepsilon \to 0$ as can be understood from figure 1. The cost on line segments is null and its value on the arc of circle $\widehat{P_1 P_2}$ is $\frac{\Delta\theta}{r}$, so if points P_1 and P_2 tend to infinity also r will and ε will tend to zero. The junctions between straight lines and the arc of the circle where curvature is not defined can be made smooth with a Cornu spiral [10] which will not affect the cost when P_1 and P_2 tend to infinity. Solutions of infinite length as the one shown in figure 1 can not be found by variational calculus as they belong to the closure of the open set of curves in $\mathbf{R}^2$. It will be demonstrated that if $|\theta(s) - \theta_0| > \pi$ holds, a finite length solution never exists, so either an additional constraint on total length must be added or the cost function must be changed in order to penalize length.

3 Variational approach solution

If a $3D$ curve $\mathbf{C}(\zeta) = (x(\zeta), y(\zeta), z(\zeta))$ is expressed in terms of a generic parameter ζ, it's square curvature is given by

$$k^2(\zeta) = \frac{\left| \frac{d\mathbf{C}}{d\zeta} \wedge \frac{d^2\mathbf{C}}{d\zeta^2} \right|^2}{\left| \frac{d\mathbf{C}}{d\zeta} \right|^6} = \frac{(y'z'' - y''z')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^3} + \tag{3}$$

$$+ \frac{(z'x'' - z''x')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^3} + \frac{(y''x' - y'x'')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^3}$$

where the symbol $'$ indicates the derivative with respect to ζ. Remembering that the length element ds can be expressed as $ds = (x'^{\,2} + y'^{\,2} + z'^{\,2})^{1/2} d\zeta$, the $3D$ cost ε

equivalent to the $2D$ one of equation (1) will be

$$\left.\begin{aligned} \varepsilon = &\int_0^{\zeta_f} \frac{(y'z''-y''z')^2}{(x'^{\,2}+y'^{\,2}+z'^{\,2})^{5/2}}\ d\zeta + \\ &+\int_0^{\zeta_f} \frac{(z'x''-z''x')^2}{(x'^{\,2}+y'^{\,2}+z'^{\,2})^{5/2}}\ d\zeta + \\ &+\int_0^{\zeta_f} \frac{(y''x'-y'x'')^2}{(x'^{\,2}+y'^{\,2}+z'^{\,2})^{5/2}}\ d\zeta \end{aligned}\right\} \tag{4}$$

The $2D$ case, which must be optimized with boundary conditions given in (2), is obtained for $z = constant \rightarrow z' = z'' = 0$. Indicating respectively with G_1, G_2 and G_3 the three integrands of equation (4) the following hold $\varepsilon = \sum_{i=1}^{3} \int_0^{\zeta_f} G_i\ d\zeta$ and

$$\sum_{i=1}^{3} G_i = k^2 \frac{ds}{d\zeta} \tag{5}$$

As each G_i is positive by definition, equation (4) will be minimized if and only if each term of (4) will be; thus the minimization conditions for a generic term G_i must be sought. Lets consider for example G_3 and the minimization of $\nu = \int_0^{\zeta_f} G_3\ d\zeta$. Putting $F \equiv G_3$ and indicating with F_y, F_x and F_z its partial derivatives with respect to y, x and z, the solution $(x(\zeta), y(\zeta), z(\zeta))$ to the minimization of $\nu = \int_0^{\zeta_f} F\ d\zeta$ has to satisfy Euler-Poisson's equations [11]

$$\left.\begin{aligned} &F_x - \tfrac{d}{d\zeta}F_{x'} + \tfrac{d^2}{d\zeta^2}F_{x''} = 0 \\ &F_y - \tfrac{d}{d\zeta}F_{y'} + \tfrac{d^2}{d\zeta^2}F_{y''} = 0 \\ &F_z - \tfrac{d}{d\zeta}F_{z'} = 0 \end{aligned}\right\} \tag{6}$$

If the total length had been fixed to L^* the optimal curve would have to satisfy (6) with fixed boundary configurations, as boundary conditions (2) for the $2D$ case, and ζ_f such that $\int_0^{\zeta_f}(x'^{\,2} + y'^{\,2} + z'^{\,2})^{1/2} d\zeta = L^*$; if, on the contrary, the total length is not fixed equation (6) must hold with fixed boundary configurations, analogue to (2) in $2D$, and with the constraint of null variation $\Delta\nu$ due to the moving boundary ζ_f. The expression of the variation $\Delta\nu$ due to the moving boundary ζ_f can be calculated extending the same techniques [11] adopted when F depends on a single function and it's first derivative, i.e. $F = F(x, y(x), y'(x))$, to the present situation where $F = F(\zeta, x, y, z, x', y', z', x'', y'', z'')$. Assuming equation (6) to be satisfied the variation due to moving boundary is

$$\begin{aligned} \Delta\nu = &\Big[F - y''F_{y''} - y'\left(F_{y'} - \tfrac{d}{d\zeta}F_{y''}\right) + \\ &-x''F_{x''} - x'\left(F_{x'} - \tfrac{d}{d\zeta}F_{x''}\right) - z'F_{z'}\Big]\Big|_{\zeta_f}\ \delta\zeta_f + \\ &+F_{y''}\big|_{\zeta_f}\ \delta y'_f + F_{x''}\big|_{\zeta_f}\ \delta x'_f + \left(F_{y'} - \tfrac{d}{d\zeta}F_{y''}\right)\Big|_{\zeta_f}\ \delta y_f \\ &+\left(F_{x'} - \tfrac{d}{d\zeta}F_{x''}\right)\Big|_{\zeta_f}\ \delta x_f + F_{z'}\big|_{\zeta_f}\ \delta z_f \end{aligned} \tag{7}$$

For fixed boundary configurations, as required by (2) in $2D$, $\delta x_f = \delta y_f = \delta z_f = \delta x'_f = \delta y'_f = \delta z'_f = 0$ and $\delta\zeta_f \neq 0$ as the final configuration is assigned, but length is not. Thus to guarantee null $\Delta\nu$ the term in square brackets of equation (7) must be null.

With reference to equation (6) note that $F_x = F_y = F_z = 0$ and $F_\zeta = 0$ by definition of F so that the following first integrals must hold:

$$\left.\begin{aligned} F_{x'} - \tfrac{d}{d\zeta}F_{x''} &= -\alpha_1 \\ F_{y'} - \tfrac{d}{d\zeta}F_{y''} &= -\alpha_2 \\ F_{z'} \qquad\quad &= -\alpha_3 \end{aligned}\right\} \tag{8}$$

for some constant α_1, α_2 and α_3. Moreover, by direct calculation follows that $F - y''F_{y''} - x''F_{x''} = -F$ and that $\frac{d}{d\zeta}(F - y''F_{y''} - x''F_{x''}) = -\frac{d}{d\zeta}F = x''[F_{x'} - \frac{d}{d\zeta}F_{x''}] + y''[F_{y'} - \frac{d}{d\zeta}F_{y''}] + z''F_{z'}$. Substituting equation (8) in this last equation and integrating implies

$$F \equiv \frac{(y''x' - y'x'')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^{5/2}} = \alpha_1\ x' + \alpha_2\ y' + \alpha_3\ z' + \beta \tag{9}$$

This differential equation must be solved with fixed boundary configurations, conditions (2) in $2D$, and either $\int_0^{\zeta_f}(x'^{\,2} + y'^{\,2} + z'^{\,2})^{1/2} d\zeta = L^*$ if L^* is fixed, or $\Delta\nu = 0$ being $\Delta\nu$ defined in (7) if maximum length is not fixed. This latter hypothesis implies $\beta = 0$ as can be shown substituting (8) in (7). The same equation (9) demonstrated for $F \equiv G_3$ can be shown to hold, with different constants α_i $i = 1, 2, 3$ and β, for G_1 and G_2, so substituting these equations in (5) the general $3D$ Euler-Poisson equation solving the optimization problem (1) with fixed boundary configurations for an arbitrary parameterization ζ is found to be:

$$k^2 \frac{ds}{d\zeta} = \mathbf{a} \cdot \frac{d\mathbf{C}(\zeta)}{d\zeta} + b \tag{10}$$

where $\mathbf{a}$ and b are constants. As follows from the above discussion, b is either null if no length constraint is imposed, or eventually non null in order to satisfy a given length L^*. As torsion is not specified, equation (10) by itself, projected on a plane and with given boundary configurations, uniquely determines a $2D$ curve, but not a $3D$ one. In the $2D$ situation $z = constant$ with a curvilinear parameterization $\zeta = s$ equation (10) is reduced to the same equations calculated in the plane starting from a Cartesian parameterization[8 12], i.e.,

$$k^2(s) = \mathbf{a} \cdot \mathbf{T}(s) + \beta = \alpha \cos(\theta - \varphi) + \beta \qquad (11)$$

being the vector $\mathbf{a} = (\alpha_1, \alpha_2)$, α it's norm and φ it's phase.

4 $2D$ Solution properties

With reference to equation (11) the following properties hold:

i) If no constraint is imposed on maximum length (i.e. $\beta = 0$, see (7)) and $|\theta(s) - \theta_0| > \pi$ for some s equation (11) has no solution other than $\alpha = 0$, i.e. a straight line of infinite length, a solution of the kind depicted in figure (1). Moreover when a finite non-constrained length solution exists ($\beta = 0$, but $\cos(\theta - \varphi) > 0$ on the whole path) it is never a finite radius circular arc (constant non null curvature) as equation (11) shows that constant curvature would imply a constant unit tangent vector $\mathbf{T}(s)$, i.e. a straight line once again.

ii) To completely determine the path from equation (11) the constants α_1, α_2 and, eventually, β must be calculated on the basis of boundary conditions (2). As suggested by M.Kallay[12] this may be accomplished solving the following nonlinear system

$$\left.\begin{aligned} x_f &= \int_{\theta_0}^{\theta_f} \frac{\cos(\theta)}{k(\theta)}\, d\theta \\ y_f &= \int_{\theta_0}^{\theta_f} \frac{\sin(\theta)}{k(\theta)}\, d\theta \\ L^* &= \int_{\theta_0}^{\theta_f} \frac{1}{k(\theta)}\, d\theta \end{aligned}\right\} \qquad (12)$$

being k given by equation (11). The initial configuration can always be thought as ($x_0 = 0, y_0 = 0, \theta_0 = 0$) as this is equivalent to choosing the reference frame. The last equation of (12) is needed to calculate β if the final length is assigned. Nevertheless from an engineering point of view fixing the total length is as unreasonable as dealing with infinitely long paths. The most natural approach is to weight curvature and length through some parameter. Indeed within the developed formulation (equations 6 through 8) it can be shown that if the cost function to be minimized is changed from equation (1) with *fixed* L to $\int_0^L (k^2 + \mu)\, ds$ with *unfixed* L, being μ a positive constant that penalizes length, the Euler-Poisson equation to be solved has exactly equation's (11) structure with the fixed μ parameter in place of the unknown β, i.e. $k^2(s) = \mathbf{a} \cdot \mathbf{T}(s) + \mu$. This is not surprising as μ (or β) can be thought of as a Lagrange multiplier that transforms the L-constrained minimization of (1) problem, in the equivalent L-unconstrained minimization of $\int_0^L (k^2 + \mu)\, ds$ problem. Given this different and more appealing interpretation of the freely fixed β it will be sufficient to solve the first two equations of (12) in order to calculate $\mathbf{a}$ and thus the optimal path.

iii) If boundary conditions (2) are such that $\theta(s) \simeq 0$ over the whole length of the path than the tangent vector $\mathbf{T}(s)$ can be approximated by $\mathbf{T}(s) \cong (1, \theta(s))$ so that equation (11) implies $\dot\theta^2(s) = \alpha_2\ \theta(s) + \beta + \alpha_1$ being $\dot\theta(s) = \frac{d\theta}{ds} = k$ by definition of curvature. Integrating this equation with initial condition $\theta(0) = 0$ yields $\theta(s) = \frac{\alpha_2}{4} s^2 \pm s\sqrt{\beta + \alpha_1}$ or

$$k(s) = \frac{\alpha_2}{2} s \pm \sqrt{\beta + \alpha_1} \qquad (13)$$

i.e., the curve is a clothoid or Cornu spiral. Cornu spirals are curves defined by $k(s) = k_c s + k_0$ and are used mostly in highway and railway design to link smoothly (up two second derivative) two curves possibly of different curvature[10] as two circles of different radius, straight lines and circles, two different straight lines, or similar. Special-case clothoids are circles ($k_c = 0, k_0 \neq 0$) and straight lines ($k_c = k_0 = 0$). In robotic applications they have been first analyzed by Kanayama et al.[13] and used for smoothing trajectories by Fleury et al.[14], but apparently had never shown to be minimal energy when $\theta(s) \simeq 0$. The major limit in their use is due to the difficulty in calculating k_c and k_0 for given boundary configurations. Nevertheless in the hypothesis $\theta(s) \simeq 0$ (the only case of interest) clothoids can be approximated by a cubic polynomial with the same degree of approximation used in $\mathbf{T}(s) \cong (1, \theta(s))$. From equation (9) when $z' \equiv z'' \equiv 0$ ($2D$) and $\zeta \to x$ (Cartesian parameterization) and approximating $(1+y'^2(x)) \sim 1\ \forall\, x$ (which is equivalent to $\mathbf{T}(s) \cong (1, \theta(s))\ \forall\ s$) follows that $y''^2(x) = \alpha_2\ y'(x) + \alpha_1 + \beta \Longrightarrow y(x) = \sum_{n=0}^{3} a_n x^n$ i.e. a cubic polynomial satisfying the two boundary configurations.

5 Results and examples

Figure 2 refers to paths optimizing $\int_0^L (k^2 + \mu)\, ds$ with unfixed L and passing through boundary configurations $(0, 0, 0)$ and $(1, 1, \frac{7}{4}\pi)$ for different values of μ. They have been numerically calculated solving equivalent forms of the first two equations of (12) for each θ. $\mathbf{a}$ has been previously estimated to match the given boundary conditions. Note that if $k(s) = 0$ for some s there may be an ambiguity in the choice of k's sign: this situation occurs whenever the path has an S shape and must be dealt with some attention. A possible solution, yet not the only and perhaps not the best one, is to fix some via-points among which curvature does not change sign.

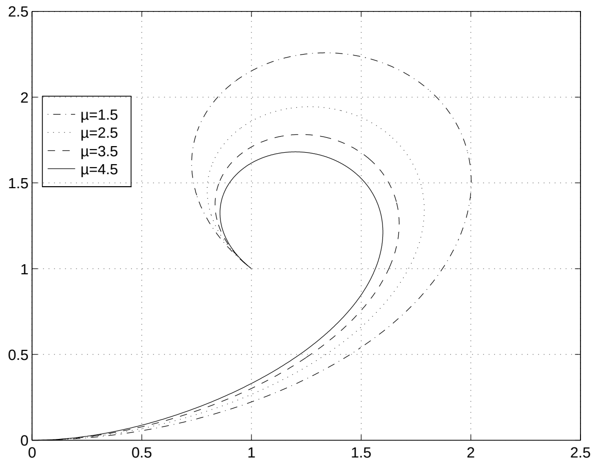

Figure 2: Optimal paths for different values of μ

6 Conclusions

The problem of finding $3D$ paths minimizing the integral of their square curvature over length with fixed boundary configurations has been addressed and solved with a variational approach for an arbitrary curve parameterization (equation (10)). It has been shown that the projection of this general solution on a plane is in agreement with the $2D$ results obtained starting from Cartesian parameterized curves by B.K.P.Horn [8] and M.Kallay [12] within computer graphics research. Underwater path planning applications have been suggested and discussed. Moreover it has been shown both, that given suitable boundary configurations the optimal path linking them is a Cornu spiral, and that in the same situation such curve is approximated by a simple cubic polynomial. The major drawback of the proposed planning method is related to the solution of the non linear system (12) in particular for S shaped paths. Future work will focus on numerical algorithms to overcome these difficulties.

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