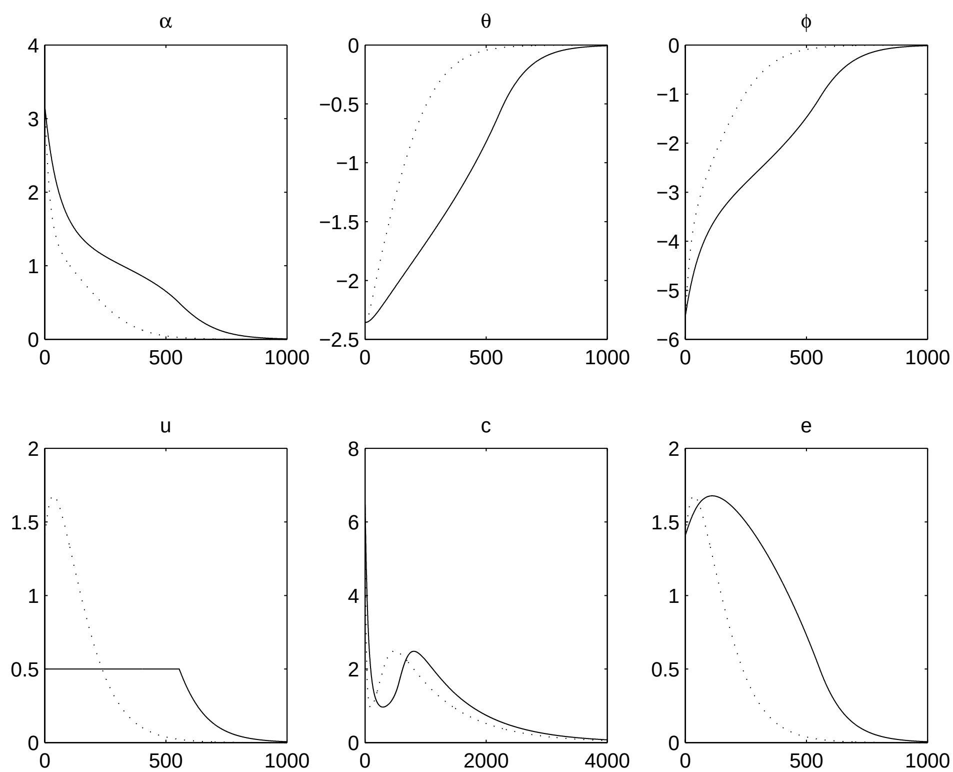
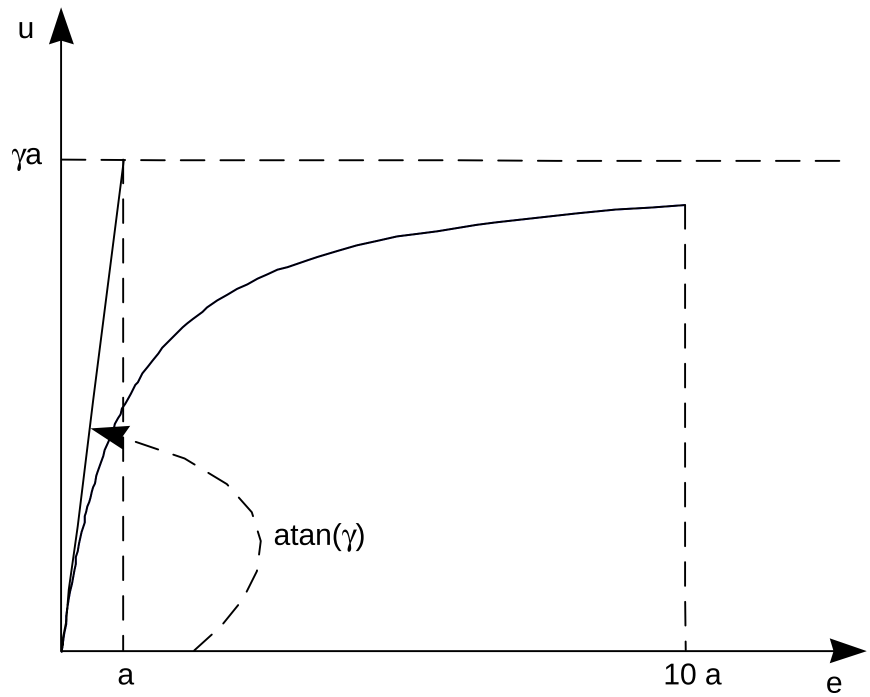


5.9.Saturated ($\bar{u} = 0.5$, solid lines) and unsaturated (dashed lines) results for the same initial configuration $(1, 1, -7\pi/4)$ and same gains $\gamma = 1$, $\beta = 2.91$, $h = 2$.



5.10.Saturated velocity input signal u.

and $\dot{V}$ gives

$$\begin{aligned}\dot{e} &= -\frac{\gamma e}{e/a+1}\cos\alpha \\ \dot{\alpha} &= -\frac{\gamma e}{e/a+1}\left(c-\frac{\sin\alpha}{e}\right) \\ \dot{\theta} &= \frac{\gamma}{e/a+1}\sin\alpha\end{aligned} \tag{5.22}$$

$$\begin{aligned}\dot{V} &= \frac{d}{dt}(\frac{1}{2}\alpha^2+h\theta^2)=\alpha\dot{\alpha}+h\theta\dot{\theta}= \\ &= \frac{\gamma}{e/a+1}(\alpha\sin\alpha+h\theta\sin\alpha-ce\alpha)\end{aligned}$$

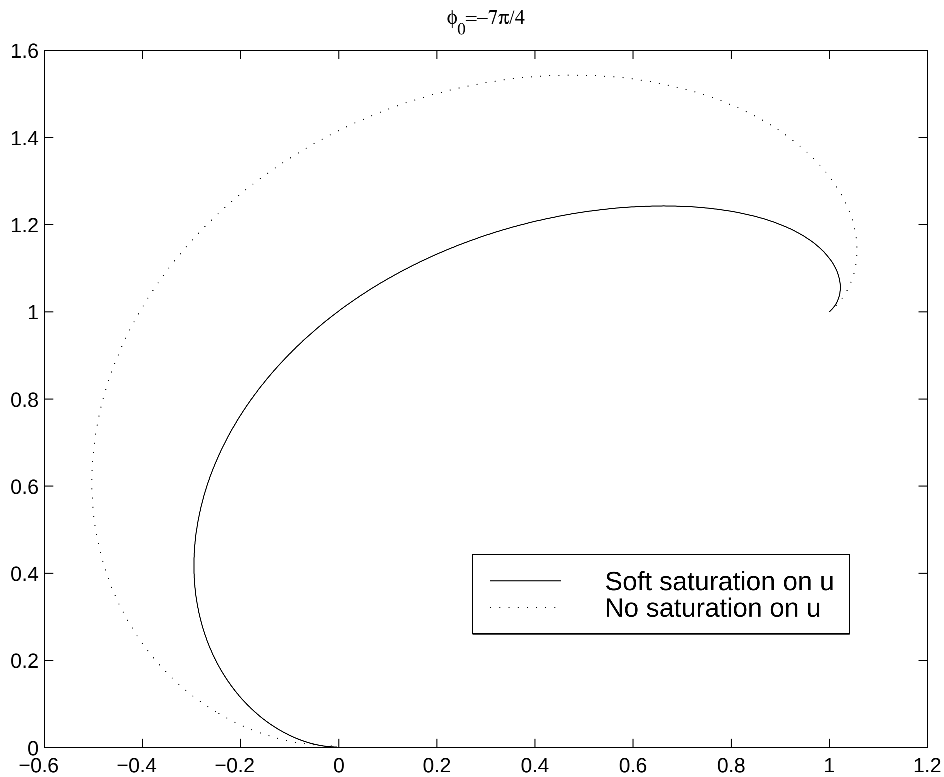
which suggests to choose c as

$$c=\left(\frac{\sin\alpha}{e}+h\frac{\theta}{e}\frac{\sin\alpha}{\alpha}+\beta\frac{\alpha}{e}\right)(e/a+1):\beta>0 \tag{5.23}$$

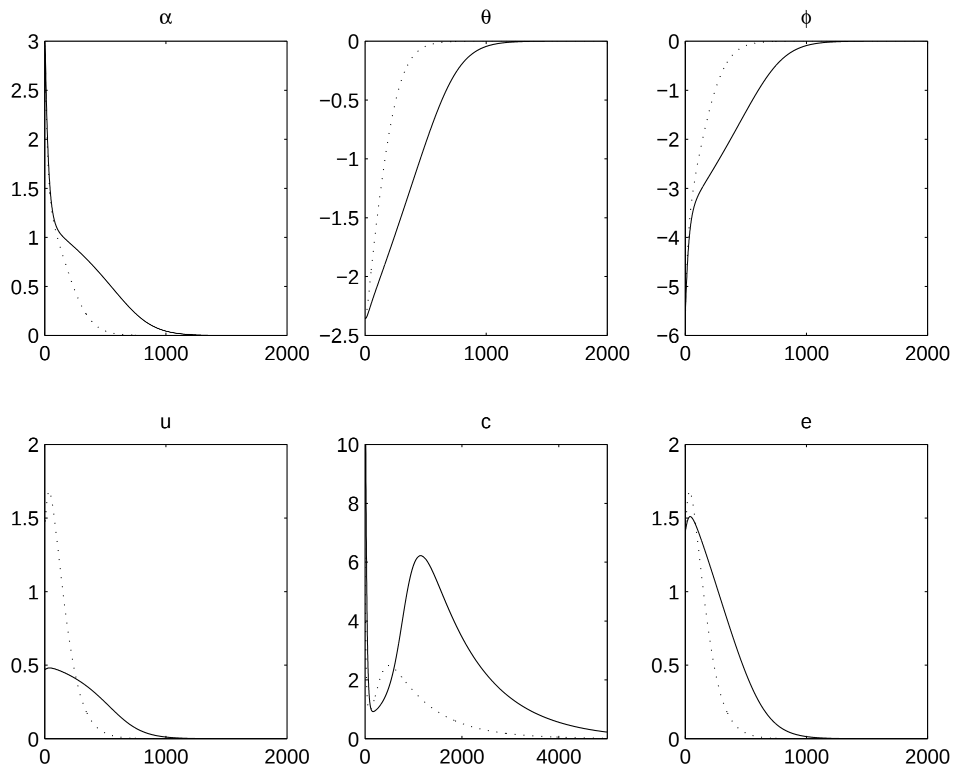
With such a choice of c the derivative of the candidate Lyapunov function V is once again given by equation (5.13) and the whole global stability analysis described above for the unsaturated $u=\gamma e$ control law can be easily replicated: the conclusion is that the system given by equation (5.7) is globally asymptotically stable under the action of the control signals u and c given by equations (5.21) and (5.23) (or equations (5.15)). Indeed when $e \ll a$ the two control strategies are identical an so are the stability properties of the system. On the contrary when $e \gg a$ the linear velocity u tends to its upper bound γa (rather then to infinity) and c tends to the finite, but not null, value of:

$$\lim_{e\to\infty} c=\frac{1}{a}\left(\sin\alpha+h\theta\frac{\sin\alpha}{\alpha}+\beta\alpha\right)$$

This shows that the a parameter must be tuned keeping into account a trade off between the maximum allowed linear speed γa and the maximum allowed curvature. Notice however that the curvature of the path generated by the controls given by equations (5.23) and (5.21) will generally be larger then the curvature relative to the unsaturated scheme: indeed the curvature given by equation (5.23) is the same one given by equations (5.15) multiplied by $(e/a+1)>1$. The bounded, smooth and "slow" dependence of u from e given by equation (5.21) is payed in terms of a larger curvature. This is clearly visible in the simulation results shown in figures (5.11) (5.12).



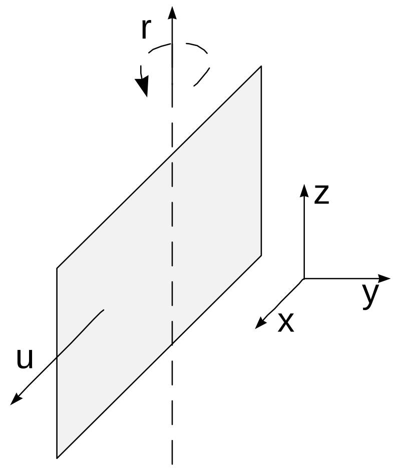
5.11.The solid and dashed line paths correspond to the implementation of the *soft* saturation scheme being $a = 1$, $\gamma = 0.8$, $h = 2$, $\beta = 2.91$ and the unsaturated scheme being $\gamma = 1$, $h = 2$, $\beta = 2.91$.



5.12.The solid and dashed lines are relative to the *soft* saturation scheme and to the unsaturated scheme of the paths shown in the previous figure.

5.2 Path Planning

Having developed a time invariant control law that globally and asymptotically stabilizes the car like systems given by equation (5.7) in the origin, attention is now focused on the path planning problem. As discussed by Aicardi et al.[2] for the unicycle model, assuming the target to move on a reference path such control scheme can be successfully employed to design a path tracking controller. The point is thus to generate a suitable reference path. In order to better define what should be considered a "suitable" reference path, some notation must be introduced. In the following a *configuration* will denote the vector describing the position and orientation at a certain time of the given system. The configuration of a $2D$ vehicle is given, e.g., by a $3D$ vector (x, y, ϕ) as shown in figure (5.1). Consider an elongated rigid body in a fluid environment as an open frame ROV, a slender body AUV, or a laminar plate grasped by a robotic manipulator moving (for simplicity) in the plane $z = 0$ (refer to figure (5.13)): as discussed in chapter 3 neglecting time varying currents the hydrodynamic load in deep water ($\geq 5m$) where wave effects are virtually absent is due to drag, lift and added mass forces. Drag is anti-parallel to the velocity and drag coefficients are proportional to the surface of attach. Lift is normal to the velocity direction, proportional to its value and to the angle of attach provided it is small enough ($\leq 12^o$ as an order of magnitude, stall occurs for higher values). Added mass forces are proportional to accelerations $\frac{du}{dt}, \frac{dv}{dt}, \frac{dr}{dt}$ through added mass coefficients which depend on the body's shape. To avoid large sway drag forces and surge added mass forces that cause major hydrodynamic load on an elongated body the lateral sway velocity v and the linear acceleration $\frac{du}{dt}$ should be kept null. Notice that the constraint on null sway makes the present problem very similar to the nonholonomic car-like path planning problem. Yaw velocity r and acceleration $\frac{dr}{dt}$ should be minimized as the large lateral surface produces strong moments along the z axis. Lift forces can be controlled through the value of surge velocity u. Thus assuming that $v = 0$ and that surge u velocity is kept constant and small to avoid added mass stresses and limit lift effects, the major dissipative force acting on the body will be caused by drag rotation moment in the z direction that at low speeds (see chapter 4) is linear in r. The energy associated with such drag moment is proportional to $\int r d\theta = \int r \frac{d\theta}{ds} ds = \int rk ds = u \int k^2 ds$ where s is the curvilinear coordinate, $k = r/u$ the paths curvature and u the constant surge velocity. This calculation suggests to consider the minimization of the cost function $J_I = \int k^2 ds$ with fixed boundary configurations as a path design criterion. Notice that such criterion produces smooth paths that are, as far as their "elastic energy" is concerned, the closest possible to a straight line. This makes the suggested criterion appealing also for wheeled land robots and indeed the problem of finding a smooth and minimum curvature trajectory between two given configurations has received a very wide attention in the robotic literature, specially regarding the steering of nonholonomic mobile robots. From the pioneering work of Dubins [83] who calculated the shortest path of bounded curvature among two configurations, many other authors focused their attention on the generation of bounded curvature $2D$ paths. In synthesis Dubins' results state



5.13.Laminar plate moving at constant z and with fixed yaw axis direction parallel to z.

that the shortest $2D$ path of bounded curvature between to fixed configurations may be traced joining straight lines and circular arcs of curvature smaller or equal to the maximum allowed. Dubins results have been extended to the case of a vehicle moving both back and forward by Reeds and Shepp [84]and more recently the issue of computing the shortest path for a nonholonomic vehicle either in an obstacle free workspace or in presence of obstacles has been discussed and refined, among the others, by J.D.Boissonnat et al.[85], X.N.Bui et al.[86], Desaulniers et al.[87], Reister et al.[88], A.M.Shkel et al.[89], Bicchi et al.[90], Moutarlier et al.[91], Desaulniers et al.[92] and Szczerba et al.[93]. Kanayama et al.[94] suggest the use of paths generated joining cubic spirals and arc of circles to minimize two cost functions related to curvature and jerk energies while A.M.Hussein et al.[95] generate smooth paths optimizing the integral of the square acceleration instead of curvature. One of the cost functions used by Kanayama et al.[94], and that is at the center of the present paper, is the integral over the path's lengths of its square curvature. A similar optimal criterion has been taken into account also by Reuter [96] within an optimal control approach. Indeed the minimization of $\int k^2 ds$ with fixed boundary configurations is a problem with an interest of its own as such cost function can be physically interpreted as proportional to the elastic energy of the curve. Due to this fact the sought plane path is sometimes called the *least energy curve* in literature. Indeed this interpretation makes the problem appealing also to researchers of other fields as A.M.Bruckstein et al.[97], B.K.P.Horn [98] and M.Kallay [99] who addressed a very similar problem to the one here discussed within a different framework and formulation. It will be shown that Horn's [98] and Kallay's [99] $2D$ results can be viewed as the projection on a plane of a more general $3D$ Euler-Poisson equation.

5.2.1 Curvature

Consider a generic differentiable curve $\mathbf{C}$ parametrized by the coordinate ξ, so that in Cartesian coordinates the points of $\mathbf{C}$ are $x = C_1(\xi)$, $y = C_2(\xi)$, $z = C_3(\xi)$. The paths curvilinear coordinate s is defined as

$$s = \int_0^{\xi} \left| \frac{d\mathbf{C}(\zeta)}{d\zeta} \right| \, d\zeta \tag{5.24}$$

being

$$\begin{aligned} \frac{d}{d\zeta}\mathbf{C} \quad &\triangleq \quad \mathbf{e}_1 \frac{dC_1}{d\zeta} + \mathbf{e}_2 \frac{dC_2}{d\zeta} + \mathbf{e}_3 \frac{dC_3}{d\zeta} \\ (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \quad &: \quad \text{reference unit vectors} \end{aligned}$$

and the unit tangent vector $\mathbf{T}$ is defined as

$$\mathbf{T} \triangleq \frac{d\mathbf{C}}{d\zeta} \Bigg/ \left| \frac{d\mathbf{C}}{d\zeta} \right| \tag{5.25}$$

Differentiating equation (5.24) it follows that

$$\frac{ds}{d\zeta} = \left| \frac{d\mathbf{C}}{d\zeta} \right| \tag{5.26}$$

so that the unit tangent $\mathbf{T}$ can be computed as

$$\mathbf{T} = \frac{d\mathbf{C}}{ds} \tag{5.27}$$

By definition the curvature is a vector given by

$$\mathbf{k} \triangleq \frac{d\mathbf{T}}{ds} \tag{5.28}$$

so that in the $2D$ case

$$\mathbf{k} \triangleq \frac{d\mathbf{T}}{ds} \Rightarrow k = \left| \frac{d\theta}{ds} \right|$$

being $d\theta$ the angular deviation relative to a step ds along the path. In many $2D$ applications the *signed curvature*

$$k = \frac{d\theta}{ds} \tag{5.29}$$

is adopted. Notice that having $\mathbf{T}$ unit constant norm, by definition, $\mathbf{k}$ and $\mathbf{T}$ are normal, i.e. $\mathbf{k} \cdot \mathbf{T} = 0$. To compute the curvature k of a generic curve $\mathbf{C} = (C_1(\xi), C_2(\xi), C_3(\xi))$

the following formula is most useful

$$k^2 = \frac{\left| \dfrac{d\mathbf{C}}{d\xi} \wedge \dfrac{d^2\mathbf{C}}{d^2\xi} \right|^2}{\left| \dfrac{d\mathbf{C}}{d\xi} \right|^6} \tag{5.30}$$

Equation (5.30) can be proved by direct calculation as follows: denoting with the symbol $'$ the derivative with respect to ξ and with m the norm of $d\mathbf{C}/d\xi$, i.e.

$$m \triangleq |\mathbf{C}'| = \left| \frac{d\mathbf{C}}{d\xi} \right|$$

equations (5.25), (5.26) and (5.28) imply

$$\mathbf{C}' = m\mathbf{T} \tag{5.31}$$
$$\mathbf{C}'' = m'\mathbf{T} + m^2\mathbf{k} \tag{5.32}$$

Next with reference to the vector property given by equation (2.3) and to the above equations for $\mathbf{C}'$ and $\mathbf{C}''$, consider

$$\begin{aligned}
\left| \frac{d\mathbf{C}}{d\xi} \wedge \frac{d^2\mathbf{C}}{d^2\xi} \right|^2 &= (\mathbf{C}' \wedge \mathbf{C}'') \cdot (\mathbf{C}' \wedge \mathbf{C}'') = ((\mathbf{C}' \wedge \mathbf{C}'') \wedge \mathbf{C}') \cdot \mathbf{C}'' = \\
&= -\mathbf{C}'' \cdot (\mathbf{C}' \wedge (\mathbf{C}' \wedge \mathbf{C}'')) = -\mathbf{C}'' \cdot (\mathbf{C}'(\mathbf{C}' \cdot \mathbf{C}'') - \mathbf{C}''(\mathbf{C}' \cdot \mathbf{C}')) = \\
&= (\mathbf{C}'' \cdot \mathbf{C}'')(\mathbf{C}' \cdot \mathbf{C}') - (\mathbf{C}' \cdot \mathbf{C}'')^2 = (m'^{\,2} + m^4k^2)m^2 - m^2m'^{\,2} = \\
&= m^6k^2
\end{aligned}$$

which concludes the proof. In terms of the Cartesian coordinates $(x(\xi), y(\xi), z(\xi))$ equation (5.30) yields

$$k^2 = \frac{(y'z'' - z'y'')^2 + (x'z'' - z'x'')^2 + (x'y'' - y'x'')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^3} \tag{5.33}$$

Moreover from equation (5.32) it follows that

$$m' = \mathbf{C}'' \cdot \mathbf{T} \Rightarrow m^2\mathbf{k} = \mathbf{C}''(\mathbf{T} \cdot \mathbf{T}) - \mathbf{T}(\mathbf{C}'' \cdot \mathbf{T}) \Rightarrow$$

$$m^2\mathbf{k} = \mathbf{T} \wedge (\mathbf{C}'' \wedge \mathbf{T}) \Rightarrow \mathbf{k} = \frac{\mathbf{C}' \wedge (\mathbf{C}'' \wedge \mathbf{C}')}{|\mathbf{C}'|^4}$$

showing the relation between curvature vector and second derivative of a curve.

5.2.2 Planning criterion: a variational calculus approach

The above discussion regarding the energy that a rigid body dissipates during a $2D$

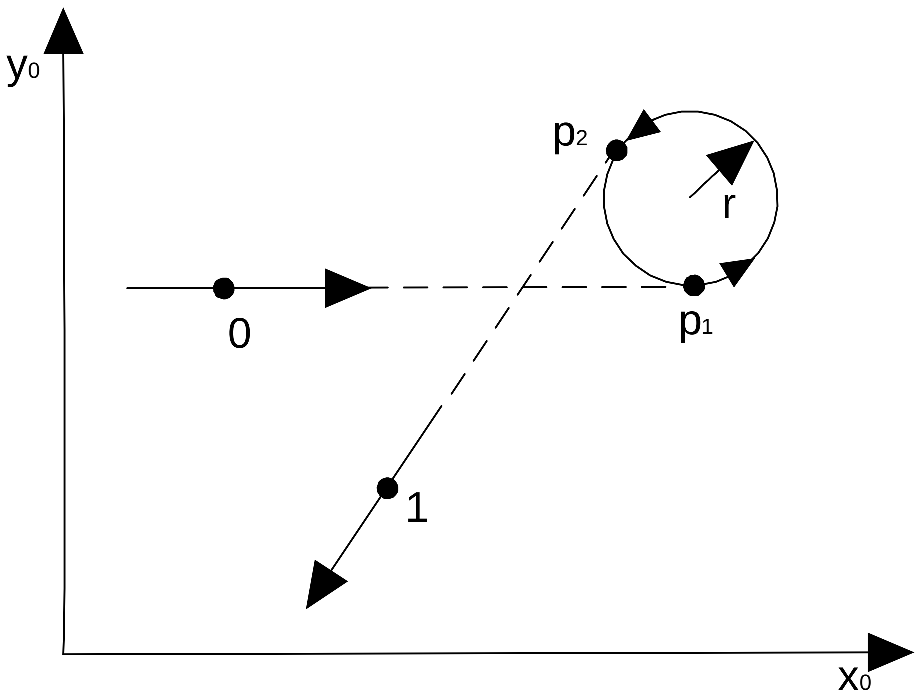
motion in a fluid suggested the minimization of the integral of the square curvature of the path over length with fixed boundary configurations, i.e.

$$J_I = \int_0^L k^2 ds \tag{5.34}$$

$$\text{in } 2D \quad \begin{cases} (x(0), y(0), \theta(0)) = (x_0, y_0, \theta_0) \\ (x(L), y(L), \theta(L)) = (x_L, y_L, \theta_L) \\ \left.\dfrac{y'}{x'}\right|_0 = \tan\theta_0 \ ; \ \left.\dfrac{y'}{x'}\right|_L = \tan\theta_L \end{cases} \tag{5.35}$$

$$\text{in } 3D \quad \begin{cases} (x(0), y(0), z(0), p(0), q(0), r(0)) = (x_0, y_0, z_0, p_0, q_0, r_0) \\ (x(L), y(L), z(L), p(L), q(L), r(L)) = (x_L, y_L, z_L, p_L, q_L, r_L) \\ p, q, r : \text{Euler angles} \end{cases} \tag{5.36}$$

Where in the $2D$ case θ_0 and θ_f are the initial and final angles between the curve and the x-axis. A most natural setting to solve the minimization of J_I with the given boundary conditions is classical analytical variational calculus which is preferred to a numerical optimal control solution as through variational calculus the general Euler-Poisson differential equation that the solution must satisfy can be computed. Notice that in equations (5.34), (5.35) and (5.36) L is not fixed and if $L \to \infty$ it is always possible to find a path for which $J_I \to 0$ as can be understood from figure (5.14). The cost on line segments is null and its value on the arc of circle $\widehat{P_1 P_2}$ is $\frac{\Delta\theta}{r}$, so if points P_1 and P_2 tend to infinity also r will and J_I will tend to zero. The junctions between straight lines and the arc of the circle where curvature is not defined can be made smooth with a Cornu spiral [100] which will not affect the cost when P_1 and P_2 tend to infinity. Solutions of infinite length as the one shown in figure (5.14) can not be found by variational calculus as they belong to the closure of the open set of curves in $\Re^2$. It will be demonstrated that if $|\theta(s) - \theta_0| > \pi$ holds for some s, a finite length solution never exists, so either an additional constraint on total length must be added or the cost function must be changed in order to penalize length. Notice that the minimization of J_I given by equation (5.34) as a planning criterion is somehow the dual problem of the most popular Dubins problem that has been extensively analyzed in the literature as discussed above. As a matter of fact the proposed planning criterion consists in finding the "least curvature" path, i.e. $\arg\min J_I$, of bounded length as opposed to the shortest path of bounded curvature, i.e. Dubins criterion. Within the nonholonomic vehicle path planning literature similar approaches have been considered by Kanayama



5.14.Infinite lenght solutions: a geometrical interpretation.

et al.[94] and Reuter [96]. In particular Kanayama et al.[94] consider the minimization of J_I given by equation (5.34), but over a given fixed length: having fixed the length it is not possible to satisfy boundary conditions as given in equations (5.35) on both the starting and ending configurations, but at most on one of the two. This is a consequence of the fact that the only solution of the minimization of J_I with fixed length, as stated in [94], is an arc of a circle [101]: with reference to equation (5.29) and indicating with $'$ the derivative with respect to the curvilinear coordinate s, i.e. $k \triangleq \frac{d\theta}{ds} \triangleq \theta'(s)$,

$$\min \int_0^{\bar{L}} \theta'^{\,2} ds : \bar{L} \text{ fixed} \Leftrightarrow \frac{\partial}{\partial\theta}\theta'^{\,2}(s) - \frac{d}{ds}\frac{\partial}{\partial\theta'}\theta'^{\,2}(s) = 0 \Rightarrow$$

$$\theta''(s) = 0 \Rightarrow \theta'(s) = const.$$

In the great majority of the practical situations length is not given a priori, but only the initial and final configurations are. Indeed if the cost J_I is to be interpreted as proportional to the "elastic energy" of the path or to the energy dissipated by rotational drag to join two given configurations, equations (5.35) must be satisfied. The cost function considered by Reuter [96] within an optimal control framework is given by

$$J_R \triangleq \int_0^L \left(\alpha k^2 + \beta \left(\frac{d^2 k}{ds^2} \right)^2 \right) ds = \alpha J_I + \beta \int_0^L \left(\frac{d^2 k}{ds^2} \right)^2 ds$$

with non-fixed length. The major advantage of considering such a cost function is that having J_R a dependance on both k and k'', boundary conditions may be imposed on the direction, the curvature and the curvature derivative at the boundary positions, i.e. the minimum of J_R must be computed adding to the boundary conditions given by equation (5.35) the conditions $k(0) = k_0$, $k(L) = k_L$, $k'(0) = k'_0$, $k'(L) = k'_L$. Nevertheless in [96] only the numerical solution of the optimization problem $\arg\min J_R$ is addressed and such solutions solves the problem of interest $\arg\min J_I$ only if $\beta = 0$, thus a variational approach solution to the minimization of J_I for a generic curve parametrization will be presented for both the $3D$ and $2D$ case.

With reference to equation (5.30) and remembering that for the arbitrary parametrization ξ the infinitesimal curve length element ds can be written as $ds = (x'^{\,2} + y'^{\,2} + z'^{\,2})^{1/2}$ being $'$ the derivative operator with respect to ξ, the cost function J_I is

$$\begin{aligned} J_I \;=\; & \int_0^{\xi_f} \frac{(y'z'' - y''z')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^{5/2}} d\xi + \int_0^{\xi_f} \frac{(z'x'' - z''x')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^{5/2}} d\xi \\ & + \int_0^{\xi_f} \frac{(y''x' - x''y')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^{5/2}} d\xi \end{aligned} \tag{5.37}$$

The $2D$ case, which must be optimized with boundary conditions given in (5.35), is obtained for $z = constant \rightarrow z' = z'' = 0$. Indicating respectively with G_1, G_2 and G_3 the three integrands of equation (5.37) the following hold $J_I = \sum_{i=1}^{3} \int\limits_0^{\xi_f} G_i \, d\xi$ and

$$\sum_{i=1}^{3} G_i = k^2 \frac{ds}{d\xi} \tag{5.38}$$

As each G_i is positive by definition, equation (5.37) will be minimized if and only if each term of (5.37) will be; thus the minimization conditions for a generic term G_i must be sought. Lets consider for example G_3 and the minimization of $\nu = \int_0^{\xi_f} G_3 \, d\xi$. Assuming

$$F \triangleq G_3 = \frac{(y''x' - x''y')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^{5/2}} \tag{5.39}$$

and indicating with F_λ its partial derivative with respect to any quantity λ the solution $(x(\xi), y(\xi), z(\xi))$ to the minimization of $\nu = \int_0^{\xi_f} F \, d\xi$ has to satisfy Euler-Poisson's equations [101]:

$$\left.\begin{aligned}
&F_x - \frac{d}{d\xi} F_{x'} + \frac{d^2}{d\xi^2} F_{x''} = 0 \\
&F_y - \frac{d}{d\xi} F_{y'} + \frac{d^2}{d\xi^2} F_{y''} = 0 \\
&F_z - \frac{d}{d\xi} F_{z'} = 0
\end{aligned}\right\} \tag{5.40}$$

If the total length had been fixed to L^* the optimal curve would have to satisfy (5.40) with fixed boundary configurations as given by equations (5.35) in $2D$ or equations (5.36) in $3D$, and ξ_f such that $\int_0^{\xi_f} (x'^{\,2} + y'^{\,2} + z'^{\,2})^{1/2} d\xi = L^*$; if, on the contrary, the total length is not fixed equation (5.40) must hold with fixed boundary configurations given by equations (5.35) or (5.36) and with the constraint of null variation $\Delta\nu$ due to the moving boundary ξ_f. The expression of the variation $\Delta\nu$ due to the moving boundary ξ_f can be calculated extending the same techniques [101] adopted when F depends on a single function and it's first derivative, i.e. $F = F(x, y(x), y'(x))$, to the present situation where $F = F(\xi, x, y, z, x', y', z', x'', y'', z'')$. Assuming equation

(5.40) to be satisfied the variation due to moving boundary is

$$\begin{aligned}\Delta\nu = &\Big[F - y''F_{y''} - y'\left(F_{y'} - \tfrac{d}{d\xi}F_{y''}\right) + \\ &-x''F_{x''} - x'\left(F_{x'} - \tfrac{d}{d\xi}F_{x''}\right) - z'F_{z'}\Big]\Big|_{\xi_f}\delta\xi_f + \\ &+F_{y''}\big|_{\xi_f}\delta y'_f + F_{x''}\big|_{\xi_f}\delta x'_f + \left(F_{y'} - \tfrac{d}{d\xi}F_{y''}\right)\Big|_{\xi_f}\delta y_f \\ &+\left(F_{x'} - \tfrac{d}{d\xi}F_{x''}\right)\Big|_{\xi_f}\delta x_f + F_{z'}\big|_{\xi_f}\delta z_f\end{aligned} \tag{5.41}$$

For fixed boundary configurations, as required by (5.35) in $2D$, $\delta x_f = \delta y_f = \delta z_f = \delta x'_f = \delta y'_f = \delta z'_f = 0$ and $\delta\xi_f \neq 0$ as the final configuration is assigned, but length is not. Thus to guarantee null $\Delta\nu$ the term in square brackets of equation (5.41) must be null. With reference to equations (5.40) and (5.39) notice that $F_x = F_y = F_z = 0$ and $F_\xi = 0$ by definition of F so that the following first integrals must hold:

$$\left.\begin{aligned} F_{x'} - \frac{d}{d\xi}F_{x''} &= -\alpha_1 \\ F_{y'} - \frac{d}{d\xi}F_{y''} &= -\alpha_2 \\ F_{z'} &= -\alpha_3 \end{aligned}\right\} \tag{5.42}$$

for some constant α_1, α_2 and α_3. Moreover, by direct calculation follows that

$$F - y''F_{y''} - x''F_{x''} = -F$$

and that

$$\begin{aligned} -\frac{d}{d\xi}F &= \frac{d}{d\xi}(F - y''F_{y''} - x''F_{x''}) = \\ &= x''[F_{x'} - \frac{d}{d\xi}F_{x''}] + y''[F_{y'} - \frac{d}{d\xi}F_{y''}] + z''F_{z'} \end{aligned}$$

Substituting equation (5.42) in this last equation and integrating implies

$$F = \frac{(y''x' - y'x'')^2}{(x'^{\,2} + y'^{\,2} + z'^{\,2})^{5/2}} = \alpha_1\,x' + \alpha_2\,y' + \alpha_3\,z' + \beta \tag{5.43}$$

This differential equation must be solved with boundary configurations given by equations (5.36) and either $\int_0^{\zeta_f}(x'^{\,2} + y'^{\,2} + z'^{\,2})^{1/2}d\zeta = L^*$ if L^* is fixed, or $\Delta\nu = 0$ being $\Delta\nu$ defined in (5.41) if maximum length is not fixed. This latter hypothesis implies $\beta = 0$ as can be shown substituting (5.42) in (5.41). Moreover equation (5.43) that has been derived for $F \equiv G_3$ can be shown to hold, with different constants α_i $i = 1, 2, 3$

and β, also for G_1 and G_2. As a consequence substituting these equations in (5.38) the general $3D$ Euler-Poisson equation solving the optimization problem (5.34) for an arbitrary parameterization ξ is found to be:

$$k^2 \frac{ds}{d\xi} = \mathbf{a} \cdot \frac{d\mathbf{C}(\xi)}{d\xi} + b \tag{5.44}$$

where $\mathbf{a}$ and b are constants that depend on the given boundary conditions. As follows from the above discussion, b is either null if no length constraint is imposed, or eventually non null in order to satisfy a given length L^*. As torsion is not specified, equation (5.44) by itself, projected on a plane and with given boundary configurations, uniquely determines a $2D$ curve, but not a $3D$ one. In the $2D$ situation $z = constant$ with a curvilinear parameterization $\xi = s$ equation (5.44) is reduced to the same equations calculated in the plane starting from a Cartesian parameterization [98] [99], i.e.,

$$k^2(s) = \mathbf{a} \cdot \mathbf{T}(s) + \beta = \alpha \cos(\theta - \varphi) + \beta \tag{5.45}$$

being the vector $\mathbf{a} = (\alpha_1, \alpha_2)$, α it's norm and φ it's phase. Equation (5.45) had been already presented by Horn [98] in 1983 and then discussed by Kallay [99] and Bruckstein et al.[97] in 1986 and 1990 within the computer graphics research community. Nevertheless in these previous works the variational problem was solved for a one valued real function $y : \Re \rightarrow \Re$ and the so computed Euler equation was then "extended" to the case of a $2D$ curvilinear parametrized curve $(x(s), y(s))$. Indeed the $2D$ result is the same, but a priori this fact is not obvious as the set of real valued functions $y(x)$ among which the solution was initially computed is a subset of the larger set of $2D$ curves $(x(\xi), y(\xi))$. Moreover having approached and solved the minimization problem directly in the family of $3D$ curves, the most general $3D$ solution given by equation (5.44) has been obtained [102] and a much deeper insight in the interpretation of the β parameter has been presented.

5.2.3 Solution properties

With reference to equation (5.45) the following properties hold:

i) If no constraint is imposed on maximum length (i.e. $\beta = 0$, see (5.41)) and $|\theta(s) - \theta_0| > \pi$ for some s equation (5.45) has no solution other than $\alpha = 0$, i.e. a straight line of infinite length, a solution of the kind depicted in figure (5.14). Moreover when a finite non-constrained length solution exists ($\beta = 0$, but $\cos(\theta - \varphi) > 0$ on the whole path) it is never a finite radius circular arc (constant non null curvature) as equation (5.45) shows that constant curvature would imply a constant unit tangent vector $\mathbf{T}(s)$, i.e. a straight line once again.

ii) To completely determine the path from equation (5.45) the constants α_1, α_2 and, eventually, β must be calculated on the basis of boundary conditions (5.35). As suggested by M.Kallay [99], if the paths curvature is strictly different from zero over the whole length, this may be accomplished solving numerically the following nonlinear system

$$\left.\begin{aligned} x_f &= \int_{\theta_0}^{\theta_f} \frac{\cos(\theta)}{k(\theta)}\, d\theta \\ y_f &= \int_{\theta_0}^{\theta_f} \frac{\sin(\theta)}{k(\theta)}\, d\theta \\ L^* &= \int_{\theta_0}^{\theta_f} \frac{1}{k(\theta)}\, d\theta \end{aligned}\right\} \tag{5.46}$$

being k given by equation (5.45). If, on the contrary, the paths curvature is null for some s as when k changes sign, equations (5.46) are not defined and a different approach must be adopted. The issue of computing the path for given boundary configurations integrating equation (5.44) will be discussed in the following section for both constant and non constant sign curvature paths. The initial configuration can always be thought as $(x_0 = 0, y_0 = 0, \theta_0 = 0)$ as this is equivalent to choosing the reference frame. The last equation of (5.46) is needed to calculate β if the final length is assigned. Notice once again that if $|\theta(s) - \theta_0| \leq \pi \;\forall\; s$ belonging to the path then the length needs not to be penalized ($\beta = 0$) and the curvature can be computed for every point of the path as $k = \pm\sqrt{\mathbf{a} \cdot \mathbf{T}}$ being the sign fixed according to the curve direction. Following the previous observation the β parameter needs to be fixed to a non null positive value only if the range of the values of $\theta(s)$ along the path is such that $\mathbf{a} \cdot \mathbf{T}$ can not stay positive for every s. Nevertheless from an engineering point of view fixing the total length is as unreasonable as dealing with infinitely long paths. The most natural approach is to weight curvature and length through some parameter. Indeed within the developed formulation (equations 5.40 through 5.42) it can be shown that if the cost function to be minimized is changed from equation (5.34) with *fixed* L to $\int_0^L (k^2 + \mu)\, ds$ with *non fixed* L, being μ a positive constant that penalizes length, the Euler-Poisson equation to be solved has exactly equation's (5.45) structure with the fixed μ parameter in place of the unknown β, i.e. $k^2(s) = \mathbf{a} \cdot \mathbf{T}(s) + \mu$. This is not surprising as μ (or β) can be thought of as a Lagrange multiplier that transforms the L-constrained minimization of (5.34) problem, in the equivalent L-unconstrained minimization of $\int_0^L (k^2 + \mu)\, ds$ problem. Given this different and more appealing interpretation of the freely fixed β it will be sufficient to solve the first two equations of (5.46) in order to calculate $\mathbf{a}$ and thus the optimal path.

iii) If boundary conditions (5.35) are such that $\theta(s) \simeq 0$ over the whole length of the path than the tangent vector $\mathbf{T}(s)$ can be approximated by $\mathbf{T}(s) \cong (1, \theta(s))$ so that equation (5.45) implies

$$\frac{d\theta}{ds} = \alpha_2\, \theta(s) + \beta + \alpha_1$$

being $\frac{d\theta}{ds} = k$ by definition of curvature. Integrating this equation with initial condition

$\theta(0) = 0$ yields $\theta(s) = \frac{\alpha_2}{4}s^2 \pm s\sqrt{\beta + \alpha_1}$ or

$$k(s) = \frac{\alpha_2}{2}s \pm \sqrt{\beta + \alpha_1} \tag{5.47}$$

i.e., the curve is a clothoid or Cornu spiral. Cornu spirals are curves defined by $k(s) = k_c s + k_0$ and are used mostly in highway and railway design to link smoothly (up two second derivative) two curves possibly of different curvature [100] as two circles of different radius, straight lines and circles, two different straight lines, or similar. Special-case clothoids are circles ($k_c = 0, k_0 \neq 0$) and straight lines ($k_c = k_0 = 0$). In robotic applications they have been first analyzed by Kanayama et al.[103] and used for smoothing trajectories by Fleury et al.[104], but apparently had never shown to be minimal energy when $\theta(s) \simeq 0$. The major limit in their use is due to the difficulty in calculating k_c and k_0 for given boundary configurations. Nevertheless in the hypothesis $\theta(s) \simeq 0$ (the only case of interest) clothoids can be approximated by a cubic polynomial with the same degree of approximation used in $\mathbf{T}(s) \cong (1, \theta(s))$. From equation (5.43) when $z' \equiv z'' \equiv 0$ ($2D$) and $\xi \to x$ (Cartesian parameterization) and approximating $(1 + y'^2(x)) \sim 1 \ \forall\ x$ (which is equivalent to $\mathbf{T}(s) \cong (1, \theta(s)) \ \forall\ s$) follows that $y''^2(x) = \alpha_2\, y'(x) + \alpha_1 + \beta \Longrightarrow y(x) = \sum_{n=0}^{3} a_n x^n$ i.e. a cubic polynomial satisfying the two boundary configurations.

5.2.4 Solution examples

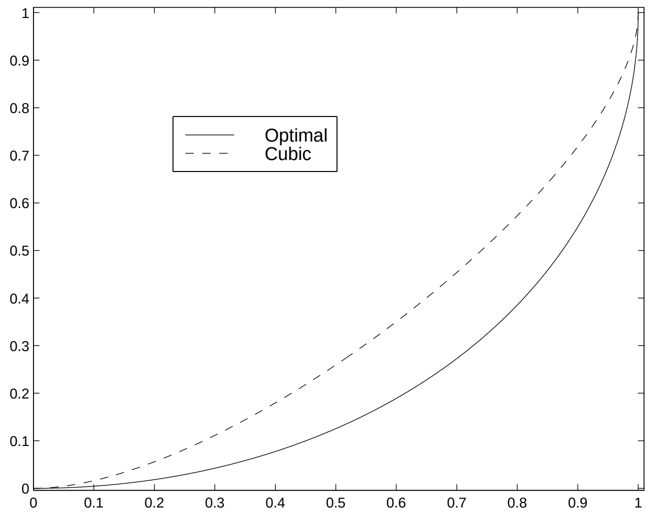
The major difficulty in the implementation of the above reported $2D$ results is related to the calculation of the parameter $\mathbf{a}$ given the final boundary configuration (as noticed previously the initial configuration can always be taken to be $(0, 0, 0)$ as this is equivalent to choosing the reference frame). Equation (5.44) can not be trivially integrated, thus a numerical algorithm is required. Two different cases may be distinguished:

- constant sign, non null curvature paths
- non constant sign curvature paths called S-*shaped* paths in the following

As far as the first case is concerned the parameters $\alpha > 0$ and φ of equation (5.45) can be computed as

$$\begin{aligned} \arg\min_{\alpha,\varphi} \Bigg[& \left(x_f - \int_0^{\theta_f} \frac{\cos\theta}{\pm\sqrt{\alpha\cos(\theta - \varphi) + \beta}} d\theta \right)^2 \\ + & \left(y_f - \int_0^{\theta_f} \frac{\sin\theta}{\pm\sqrt{\alpha\cos(\theta - \varphi) + \beta}} d\theta \right)^2 \Bigg] \end{aligned} \tag{5.48}$$

being (x_f, y_f, θ_f) the given final boundary configuration and $\beta \geq 0$ a constant that needs to be non null only if for the given (x_f, y_f, θ_f) no solution exists for $\beta = 0$, as when $|\theta_f| > \pi$. The sign in front of the square roots is unambiguously fixed according



5.15.Optimal 90^o curve, obtained assuming $(1, 1, \pi/2)$ as final configuration and $\beta = 0$, and cubic spline 90^o curve.

to the desired curve direction. The minimum problem given by equation (5.48) can be solved by standard numerical methods as the simplex method. Examples of the paths obtained with this approach are displayed in figures (5.15) and (5.17). The $\mathbf{a}$ parameter of S-shaped paths can not be computed with the above suggest method as by definition of S-shaped path the curvature takes a null value for some value θ^* of θ. As shown by the example reported in figure (5.18), where $\beta = 0$ for the sake of simplicity, once that the curvature reaches a null value as θ evolves k has to change sign as not so doing would imply a discontinuity in the derivative of k with respect to θ. Indeed once that the curvatures sign is fixed at the starting configuration $(0, 0, 0)$, the apparent possible ambiguity in ks sign choice is completely solved by the above observation: if a θ^* such that $k(\theta^*) = 0$ is reached the curvature changes sign. In order to compute $\mathbf{a}$ for a given final configuration and with reference to equations (5.6) consider the kinematics of an ideal point following the S-shaped path with unit velocity

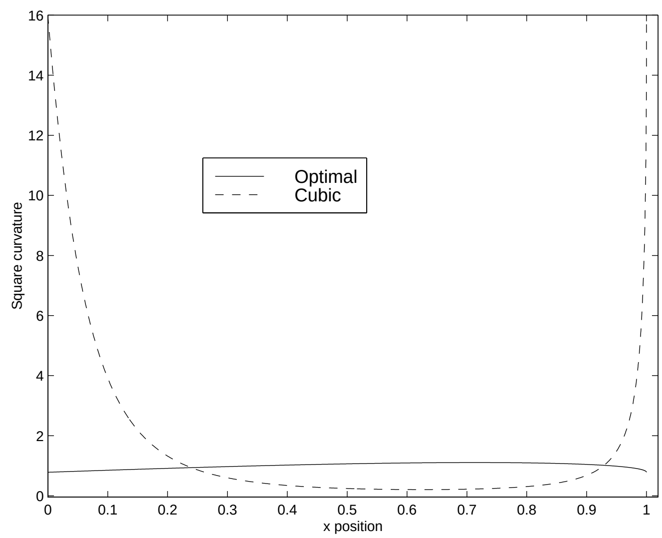
$$\dot{x} = \cos\theta \tag{5.49}$$

$$\dot{y} = \sin\theta \tag{5.50}$$

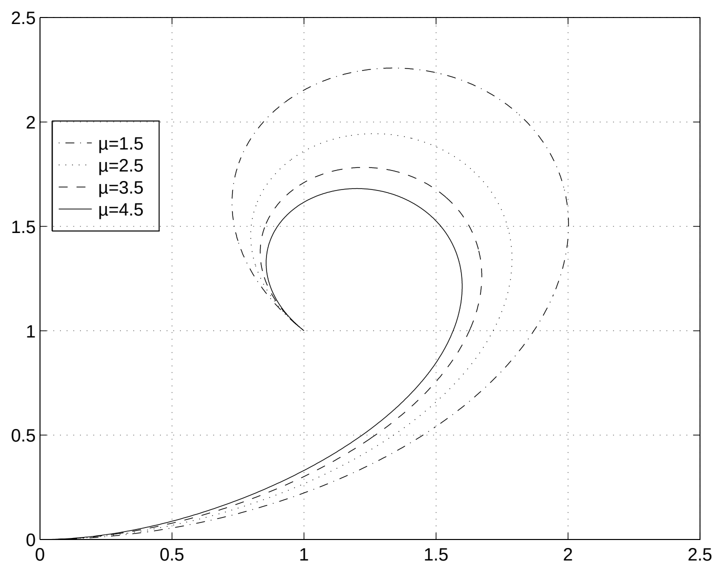
$$\dot{\theta} = k = \pm\sqrt{a_1\cos\theta + a_2\sin\theta} \tag{5.51}$$

A possible algorithm to compute $\mathbf{a} = (a_1, a_2)$ is

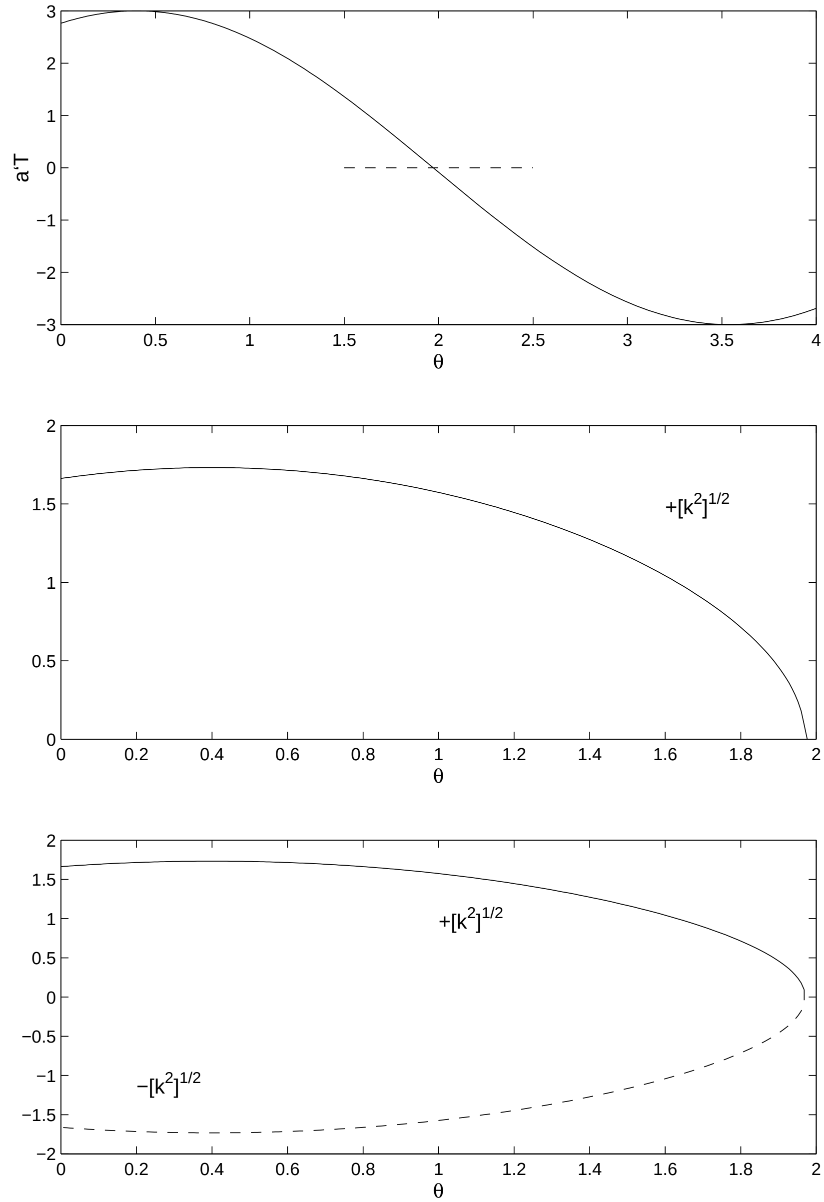
$$\mathbf{a} = \arg\min_{a_1, a_2} J_S$$



5.16.Square curvature for an optimal 90^o curve and a cubic spline path versus the x position.



5.17.Paths of constant curvature sign for final configuration $(1, 1, 7\pi/4)$ and varius values of μ, i.e. β.



5.18.From top to bottom: $k^2 = \mathbf{a} \cdot \mathbf{T}$ and k as functions of θ in the hypothesis that k is always positive or that it changes sign at θ^* being θ^*such that $k(\theta^*) = 0$.