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Modelling and Identification of Underwater Robotic Systems

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“Modelling and Identification of Underwater Robotic Systems”

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Quel che è detto è detto.
Ma sarà poi vero? Io non ho accesso
al vero, il mio pensiero ha un andamento
incerto, è sottoposto al vento
di scirocco, ma so per certo
che questi giorni invernalprimaverili
sono un eccesso inutile di luce e a me
non è concesso che attraversare i ponti
e al rosso del semaforo guardare con invidia
qualche ossesso che tra bestemmie e insulti
a passo lento infrange l'armata compatta
delle macchine. E basta, non c'è che questo.

(*Patrizia Cavalli*,
POESIE (1974-1992)
Giulio Einaudi Editore, 1992)

ABSTRACT

Whatever is the strategy pursued to design a control system or a state estimation filter for an underwater robotic system the knowledge of its identified model is very important. As far as ROVs are concerned the results presented in this thesis suggest that low cost on board sensor based identification is feasible: the detailed analysis of the residual least square costs and of the parameter estimated variances show that a decoupled vehicle model can be successfully identified by swimming pool test provided that a suitable identification procedure is designed and implemented. A two step identification procedure has been designed on the basis of: *(i)* the vehicle model structure, which has been deeply analyzed in the first part of this work, *(ii)* the type of available sensors and *(iii)* the actuator dynamics. First the drag coefficients are evaluated by constant speed tests and afterwards with the aid of their knowledge a sub-optimal sinusoidal input thrust is designed in order to identify the inertia parameters. Extensive experimental activity on the ROMEO ROV of CNR-IAN has shown the effectiveness of such approach. Moreover it has been shown that the standard unmanned underwater vehicle models may need, as for the ROMEO ROV, to take into account propeller-propeller and propeller-hull interactions that have a most relevant influence on the system dynamics (up to 50% of efficiency loss in the applied thrust with respect to the nominal model). It has been shown that such phenomena can be correctly modelled by an efficiency parameter and experimental results concerning its identification on a real system have been extensively analyzed. The parameter estimated variances are generally relatively low, specially for the drag coefficients, confirming the effectiveness of the adopted identification scheme. The surge drag coefficients have been estimated relatively to two different vehicle payload configurations, i.e. carrying a plankton sampling device or a Doppler velocimeter (see chapter 4 for details), and the results show that in the considered surge velocity range ($|u| < 1m/s$) the drag coefficients are different, but perhaps less then expected. Moreover it has been shown that in the usual operating yaw rate range ($|\dot{\psi}| < 10\deg/s$) drag is better modeled by a simple linear term rather then both a linear and a quadratic one. This is interesting as it suggests that the control system of the yaw axis of slow motion open frame ROV can be realized by standard linear control techniques. For a detailed description of the identification procedure and of the identification results of the ROMEO ROV consult chapter 4.

In the last part of this thesis the issue of planar motion control of a nonholonomic vehicle has been addressed. Inspired by the previous works of Casalino et al.[1] and Aicardi et al.[2] regarding a unicycle like kinematic model, a novel globally asymptotically convergent smooth feedback control law for the point stabilization of a car-like robot has been developed. The resulting linear velocity does not change sign, curvature is bounded and the target is asymptotically approached on a straight line. Applications to the control of underwater vehicles are discussed and extensive simulations are performed in order to analyze the algorithms behaviour with respect to actuator saturation. It is analytically shown that convergence is achieved also in presence of actuator satu-

ration and simulations are performed to evaluate the control law performance with and without actuator saturation. Moreover the generation of smooth paths having minimum square curvature, integrated over length, is addressed and solved with variational calculus in $3D$ for an arbitrary curve parametrization. The plane projection of such paths are shown to be least yaw drag energy paths for the $2D$ underwater motion of rigid bodies.

Chapter 1
Introduction

The scope of this chapter is to describe the motivations and objectives of this work.

1.1 Motivations and Objectives

Underwater robotics applications have extensively grown in the last twenty years both for scientific investigations and industrial needs. Technological improvements in the design and development of the mechanics and electronics of the systems have been followed by the development of very efficient and elaborate control strategies. Indeed the framework of underwater robotics is challenging form both a theoretical and experimental point of view. From a robotics perspective the challenge consists in dealing with an unknown parameter, highly nonlinear and coupled plant affected by non predictable noise, e.g. currents, with only partial state feedback provided by noisy and low sampling frequency sensors. This setting affects not only the *control* system synthesis, but also the *navigation* and *guidance* ones. Following [3], the navigation system is defined to be a velocity and position estimation module, the guidance system is a subsystem required to perform navigation system and, eventually, inertial reference trajectory data processing to compute local velocity and/or position references and the control system is a subsystem that takes care of generating the actuator inputs on the basis of the guidance system output. Within the classical control literature the above three subsystems are, roughly speaking, equivalent to the sensing system, the reference generator, sometimes called high level control, and the compensator (low level control).

The interest of the theoretical control system community towards underwater robotics is confirmed by the large and growing number of scientific publications and conferences touching every branch of the field. This research activity has made the state of the art in the navigation, control and guidance of underwater systems wide and variegate. As far as the control synthesis problem is concerned, all sorts of approaches have been analyzed: optimal control, adaptive control, sliding mode control, feedback linearization based control, Lyapunov based robust control, gain scheduling control, neurofuzzy and neural control. Sliding mode control for robust underwater vehicle trajectory tracking has been first proposed by the pioneer work of Yoerger and Slotine [4] in 1985. Since then many other contributions based on sliding mode control theory applied to the control of unmanned underwater vehicles (UUVs) have been proposed: among the many others, Cristi et al.[5] have reported and adaptive sliding mode approach combined with a state observer algorithm, Healey et al.[6] have discussed a multivariable sliding mode technique based on state variable errors, rather then output errors as accounted in [5], da Cunha et al.[7] have proposed a variable structure algorithm requiring only position measurements, Corradini et al.[8] have discussed a MIMO (multi input multi output) discrete time variable structure approach and Bartolini et al.[9] have suggested a second order sliding mode technique. Also adaptive control approaches for the control of UUVs have been analyzed as shown, for example, in the works of Fossen et al.[10] [11], Ramadorai et al.[12], Sagatun et al.[13] and Yuh [14]. Examples of Lyapunov based and H_∞ robust control approaches for the synthesis of underwater vehicle control systems are given by Conte et al.[15] [16] [17], while examples of the use of neural net and neurofuzzy techniques for the control of underwater vehicles are given by the works of J. Yuh[18] [19] and of Craven et al.[20]. A similarly broad range of techniques have been proposed for the synthesis of control systems for mobile base underwater manipulators. This topic is very interesting as the hydrodynamic interactions between the manipulator and the fluid may induce relevant forces on the manipulator base that should be taken into account by the system model, rather then considered external disturbances, in order to achieve satisfactory control performance. In particular, the problem of coordinated manipulator-vehicle modelling and control has been addressed, e.g., by Mahesh et al.[21], Schjølberg et al.[22], McMillan et al.[23], Tarn et al.[24], McLain et al.[25], Dunnigan et al.[26] and Canudas de Wit et al.[27].

Each of the above reported control approaches for either vehicles, manipulators or combined vehicle-manipulators systems require at some stage the knowledge of the system model and parameters. Each of the above approaches is at some extent capable of dealing with model uncertainties and system noise, but each of them necessarily needs the knowledge of a fully identified, perhaps simplified, *nominal* model. Each of the above reported control approaches increases its performance as the model uncertainty is reduced. These may seem obvious considerations that apply to any robotic system, not only to underwater ones. Indeed if complex land or space robots, e.g. manipulators, need to be identified experimentally in order to develop a reliable dynamic model, the urge for system identification applied to underwater systems is even higher as for the great majority of underwater robots model parameters can not be estimated *a priori* on the basis of geometrical or structural information. The point is that given an underwater bluff body system of known geometry, what will be its drag coefficients or its inertia parameters? There is no reliable method of answering this question without experimental data. As far as underwater vehicles are concerned, experimental data for identification can be collected either in towing tank facilities or with on board sensors. The first method relies on consolidated naval engineering methodology and is more precise but complex, lengthy and expensive. As underwater vehicles configuration is time and mission dependent, system identification by means of on board sensors is certainly more appealing being faster, cheaper and easier to be repeated for different configurations when necessary.

Another important motivation for the analysis of underwater system modelling and identification is related to state and, eventually, environment estimation problem. As pointed out at the beginning of the above discussion, underwater systems sensors generally have a low sampling rate frequency (typically less then $5Hz$ for sonar profilers and Doppler effect velocimeters) and do not provide full state feedback as not all the degrees of freedom are measured. The angular positions and, eventually, velocities are measured by inertial devices and a compass for yaw, while position with respect to the environment is measured by means of acoustic devices as long base line (LBL) or ultra short base line (USBL) positioning systems, or by sonar profilers. If velocity measurements are absent state estimation techniques as Kalman filters (KF) or extended Kalman filters (EKF) are generally adopted for velocity estimation. Indeed within this framework the need of an identified system model is related not only to control system design as discussed above, but also to the navigation one. Examples of dynamic model based navigation and motion estimation filters are given by the works of Caccia et al.[28] [29] [30] and Smith et al.[31]. The use of a correctly identified and reliable model to design dynamic filters for state estimation indirectly affects also the control system performance if the control strategy uses the estimated state as feedback. These considerations have motivated the majority of the work presented in this thesis: the development of a physical based model and its on board sensor based identification strategy for an open frame ROV. The proposed approach has been tested on the ROMEO ROV of the Institute for Naval Automation of the Italian National Research Council CNR-IAN and the experimental results are reported in this work. The proposed model is based on the classical Newton Euler unmanned underwater vehicle model presented, among others, by Yuh [14] and Fossen [32]. It is experimentally shown that such models may need to be extended in order to take into account propeller propeller, propeller hull and momentum drag interactions that are usually neglected. A two step procedure is proposed for the identification of a simplified model of the vehicles model: first the drag coefficients are estimated by constant velocity tests, then the drag coefficients values are adopted to design a suboptimal experiment for the identification of the inertia parameters.

Given the vehicles model, the motion control problem is addressed in the last part of this research and a novel algorithm for nonholonomic vehicle control taking into account the paths curvature is proposed in the last part of this work.

1.2 Outline of the work

The first chapter is mainly devoted to the discussion of the motivations and objectives of this research. In chapters 2 the adopted vector notation and some general (classical) kinematic results are presented, while in chapter 3 the dynamics of a rigid body in a fluid media is described within a Newton Euler formulation and the general equations of motion of an underwater vehicle are derived and discussed in detail. General considerations regarding underwater manipulators are also briefly addressed. Chapter 4 is devoted to the presentation of the proposed identification scheme, within the setting of classical least squares (LS) approach, and of the experimental results. At last Chapter 5 addresses the issue of nonholonomic vehicle motion control with reference to the case of underwater vehicles. Some original results regarding possible navigation solutions are presented.

1.3 Acknowledgments

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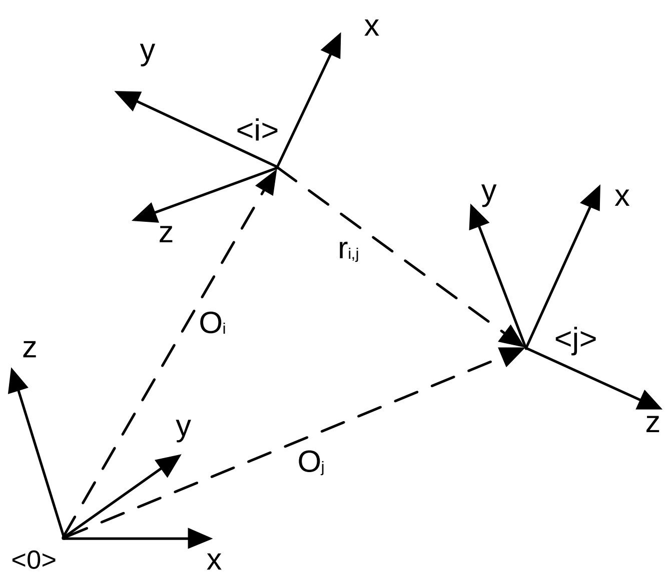
Chapter 2
Kinematics

The scope of this chapter is to introduce the adopted notation and to review some basic concepts of kinematics that will be employed.

2.1 Vectors

2.1.1 Vector notation

Free vectors will be denoted with bold characters and no particular superscript or subscript, e.g. $\mathbf{a}$, while geometric vectors, i.e. vectors projected on a specific reference frame, will be bold variables having a left hand side superscript denoting the reference frame, e.g. ${}^{i}\mathbf{a}$. The position vector $\mathbf{r}_{q,p}$ of point p with respect to point q will be written, according to Grassman's notation, as $\mathbf{r}_{q,p} = p - q$, so if O_i and O_j are the origins of reference frames $< i >$ and $< j >$ then $\mathbf{r}_{i,j} = O_j - O_i$ will denote O_js position with respect to O_i. The projection of $\mathbf{r}_{i,j}$ on reference $< n >$ is ${}^{n}\mathbf{r}_{i,j} = (\mathbf{r}_{i,j})_x\mathbf{e}_1 + (\mathbf{r}_{i,j})_y\mathbf{e}_2 + (\mathbf{r}_{i,j})_z\mathbf{e}_3 = \sum_{h=1}^{3}(\mathbf{r}_{i,j})_h\mathbf{e}_h$ being $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ an orthonormal basis of $< n >$.



2.1.2 Time derivatives of vectors

The majority of books on Robotics start with a note on reference frames, rotations and homogenous transformations. Indeed these concepts are the basis of kinematics and rely on the idea of time derivative of a vector. This is a tricky topic that is worthwhile discussing in some detail as a starting point. The building block of classical mechanics is the concept of *event*. This is the mathematical abstraction of a primitive idea that can be defined only heuristically as the limit for null time duration and space occupation of a certain physical phenomenon as viewed by an observer. The set of all events is said *space-time*, denoted by V_4 that can be identified with the Cartesian product $E_3 \times \Re$ being E_3 the $3D$ Euclidean space and $\Re$ the set of real numbers. The evolution of a material point can be described by a continuous curve (*line of universe)* in V_4 made by the sequence it's events. Given the line of universe *absolute time* can be unambiguously defined through the following:

Axiom of Absolute Time Given two events $a, b \in V_4$ their time separation $\Delta t(a, b)$ is unambiguously defined for every observer as a continuous function $\Delta t : V_4 \times V_4 \rightarrow \Re$ satisfying

$$\begin{array}{l} \Delta t(a,a) = 0 \\ \Delta t(a,b) + \Delta t(b,c) = \Delta t(a,c) \; \forall \; a,b,c \in V_4 \end{array}$$

so that chosen a reference event 0 the absolute time can be defined as the continuous function $t : V_4 \rightarrow \Re, \; t(a) \triangleq \Delta t(0, a)$. According to the properties of Δt and to the definition of $t(a)$ it follows that $\Delta t(a, b) = t(b) - t(a)$ showing the independence of Δt from the reference event. Events a and b are said to be *simultaneous* if and only if $\Delta t(a, b) = 0$.

As a consequence of the above axiom given $a \in V_4$ the equation $t = t(a)$ defines a $3D$ hyperplane Σ_t (*hyperplane of simultaneity*) subset of V_4 made of all and only the simultaneous events of a. At each fixed instant t, Σ_t can be identified with the physical space at time t, common to every observer. In particular the following axiom is assumed to hold:

Axiom of Absolute Space Each hyperplane Σ_t for a fixed $t \in \Re$, is a $3D$ space having intrinsic Euclidean structure, i.e., in every hyperplane Σ_t a Euclidean distance is defined and all axioms and theorems of Euclidean geometry hold.

It follows that every geometric result at each fixed instant has an absolute character, i.e., is independent from the observer: for example calling $V_3(\Sigma_t)$ the set of all geometric vectors at instant t the time dependent vector $\mathbf{u}(t)$ is defined as $\mathbf{u}(t) : \Re \rightarrow V_3(\Sigma_t)$ and is an absolute quantity. Notice, however, that at each instant $t_i \neq t_j$ $\mathbf{u}(t_i)$ and $\mathbf{u}(t_j)$ are elements of different spaces as Σ_{t_i} and Σ_{t_j} are not only different, but disjoint. As a matter of fact the absolute time and space axioms do not specify what is meant by a fixed point at different times and thus the same concept of movement can not be defined. In particular as $\mathbf{u}(t_i) \in V_3(\Sigma_{t_i})$ and $\mathbf{u}(t_j) \in V_3(\Sigma_{t_j})$ with $V_3(\Sigma_{t_i}) \neq V_3(\Sigma_{t_j})$

vectors $\mathbf{u}(t_i)$ and $\mathbf{u}(t_j)$ can't be compared and the incremental ratio $\frac{\mathbf{u}(t+\Delta t)-\mathbf{u}(t)}{\Delta t}$ has no meaning whatsoever. It follows that even if the concept of vector as a function of time is well posed and has an absolute meaning, it is impossible to formally introduce the time derivative of a vector on the only basis of the axioms of absolute time and space. So, as the concept of event has an absolute meaning, the one of movement and time derivative of a vector is intrinsically relative, it can not even be formally defined prior to the introduction of the concepts of *reference space* and *reference frame*. Each observer maps the space-time set V_4 in it's own distinct $3D$ Euclidean *reference space* Γ_3 with a mapping function $\pi : V_4 \rightarrow \Gamma_3$ that, according to the axiom of absolute space, must have an invertible and isometric restriction on each Σ_t; that is $\pi|_{\Sigma_t} : \Sigma_t \rightarrow \Gamma_3$ is invertible and isometric so that at each instant every observer has it's own, but coherent to all the other observers, view of the common absolute space. Given an observer 0 and it's reference space Γ_3, it's *reference frame* $< 0 >$ is an orthonormal set of 3 constant vectors in $V(\Gamma_3)$, being $V(\Gamma_3)$ the set of all geometric vectors in Γ_3. Notice that constant vectors in reference $< 0 >$ are generally time dependent as viewed by a different observer as each observer has it's own mapping π. This is the reason why time derivatives of vectors are relative to a specific observer and not absolute quantities. In particular calling $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the unit vector of reference $< 0 >$ each time dependent vector in $V(\Gamma_3)$ can be thought of as ${}^0\mathbf{u}(t) = \sum_{i=1}^{3} u_i(t)\ \mathbf{e}_i$ and the time derivative of $\mathbf{u}$ with respect to reference $< 0 >$ is $\frac{d_{<0>}}{dt}\ {}^0\mathbf{u}(t) = \sum_{i=1}^{3} \dot{u}_i(t)\ \mathbf{e}_i$ where the dot indicates the usual observer independent time derivative of a scalar function, $\dot{u}(t) \triangleq \frac{d}{dt}u(t)$. In the view of a different moving observer, say 1, the vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ may not be constant with respect to his reference frame $< 1 >$, so the time derivative of $\mathbf{u}$ with respect to $< 1 >$ is $\frac{d_{<1>}}{dt}\ {}^1\mathbf{u}(t) = \sum_{i=1}^{3}(\dot{u}_i(t)\ \mathbf{e}_i + u_i(t)\ \frac{d_{<1>}}{dt}\mathbf{e}_i)$. To better understand the nature of the term $\frac{d_{<1>}}{dt}\mathbf{e}_i$ remember that each mapping π must be isometric and invertible, so that orthonormality among vectors is observer independent. This fact is at the basis of *Poisson' s Formula*.

Poisson Formula Having noticed that orthonormal vectors in a reference frame must be viewed as orthonormal in each other, and indicating with $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ an orthonormal set of vectors fixed to reference $< 0 >$, the following holds:

$$\frac{d_{<1>}}{dt}(\mathbf{e}_i \cdot \mathbf{e}_j) = \left(\frac{d_{<1>}}{dt}\mathbf{e}_i\right) \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \left(\frac{d_{<1>}}{dt}\mathbf{e}_j\right) = \dot{\mathbf{e}}_i \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \dot{\mathbf{e}}_j = 0 \tag{2.1}$$

where $\dot{\mathbf{e}}_i \triangleq \frac{d_{<1>}}{dt}\mathbf{e}_i$. Next the time dependent $\boldsymbol{\omega}_{0/1}(t)$ vector is defined as

$$\begin{cases} \boldsymbol{\omega}_{0/1}(t) \triangleq \dfrac{1}{2}\displaystyle\sum_{h=1}^{3} \mathbf{e}_h \wedge \dot{\mathbf{e}}_h \\ \dot{\mathbf{e}}_i \triangleq \dfrac{d_{<1>}}{dt}\mathbf{e}_i \\ \mathbf{e}_h : h = 1, 2, 3 \text{ orthonormal basis of reference } < 0 > \end{cases} \tag{2.2}$$

being $\wedge$ the vector product. Remembering that for any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{2.3}$$

being $\cdot$ the scalar product, the following is calculated

$$\begin{aligned} \boldsymbol{\omega}_{0/1} \wedge \mathbf{e}_i &= -\frac{1}{2}\sum_{h=1}^{3} \left[\mathbf{e}_i \wedge (\mathbf{e}_h \wedge \dot{\mathbf{e}}_h)\right] = -\frac{1}{2}\sum_{h=1}^{3} \left[\mathbf{e}_h \left(\mathbf{e}_i \cdot \dot{\mathbf{e}}_h\right) - \dot{\mathbf{e}}_h \left(\mathbf{e}_i \cdot \mathbf{e}_h\right)\right] = \\ &= +\frac{1}{2}\sum_{h=1}^{3} \left[\mathbf{e}_h \left(\dot{\mathbf{e}}_i \cdot \mathbf{e}_h\right) + \dot{\mathbf{e}}_h \, \delta_{ih}\right] = \dot{\mathbf{e}}_i \end{aligned} \tag{2.4}$$

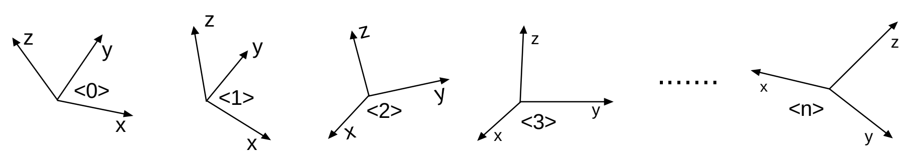
where δ_{ih} is the Kronecker symbol and the substitution $\mathbf{e}_h \left(\dot{\mathbf{e}}_i \cdot \mathbf{e}_h\right) = -\mathbf{e}_h \left(\mathbf{e}_i \cdot \dot{\mathbf{e}}_h\right)$ is possible due to equation (2.1). The equation

$$\frac{d_{<1>}}{dt}\, \mathbf{e}_i \triangleq \dot{\mathbf{e}}_i = \boldsymbol{\omega}_{0/1} \wedge \mathbf{e}_i \tag{2.5}$$

is known as Poisson's equation and allows to express the time derivative of a vector with respect to a given reference in terms of it's derivative with respect to a different reference. To stress it's physical meaning, the angular velocity vector $\boldsymbol{\omega}$ of reference $< 0 >$ with respect to the fixed reference $< 1 >$ will be denoted as $\boldsymbol{\omega}_{0/1}$. Equation (2.2) can not be considered a *definition* of angular velocity, but rather the mathematical demonstration of the existence of a free vector $\boldsymbol{\omega}$ that depends on the only relative motions of two given frames and allows to calculate the time derivative of a vector with respect to a reference as a function of the time derivative of the same vector with respect to the other reference. Remembering that $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are an orthonormal set of vectors fixed to reference $< 0 >$, the time derivative of a vector with respect to a given reference $< 1 >$ will be:

$$\begin{aligned} \frac{d_{<1>}}{dt}\, \boldsymbol{\rho} &= \frac{d_{<1>}}{dt}\, {}^0\boldsymbol{\rho} = \frac{d_{<1>}}{dt} \left(\sum_{i=1}^{3} \rho_i \mathbf{e}_i\right) = \qquad (2.6) \\ &= \sum_{i=1}^{3} \left(\frac{d_{<1>}}{dt}\, \rho_i\right) \mathbf{e}_i + \sum_{i=1}^{3} \left(\frac{d_{<1>}}{dt} \mathbf{e}_i\right) \rho_i = \\ &= \frac{d_{<0>}}{dt}\, {}^0\boldsymbol{\rho} + \boldsymbol{\omega}_{0/1} \wedge\, {}^0\boldsymbol{\rho} \qquad \Rightarrow \\ \Rightarrow \quad \frac{d_{<1>}}{dt}\, \boldsymbol{\rho} &= \frac{d_{<0>}}{dt}\, \boldsymbol{\rho} + \boldsymbol{\omega}_{0/1} \wedge\, \boldsymbol{\rho} \qquad (2.7) \end{aligned}$$

In the above calculations $\sum_{i=1}^{3} \left(\frac{d_{<1>}}{dt}\, \rho_i\right) \mathbf{e}_i$ has been replaced by $\frac{d_{<0>}}{dt}\, {}^0\boldsymbol{\rho}$ because by definition a scalar ρ is invariant for rotations, i.e. $\frac{d_{<i>}}{dt}\rho = \frac{d_{<j>}}{dt}\rho \ \forall < i >, < j >$. Notice that in (2.6) and (2.7) the geometric vector ${}^0\boldsymbol{\rho}$ has been replaced by the free vector



2.1.Kinematic chain

$\boldsymbol{\rho}$ as the time derivative of a vector depends on the reference in which it is evaluated but not on the frame eventually used to represent the vector itself. From equation (2.7) some properties of angular velocity can be deduced:

$$\frac{d_{<1>}}{dt}\,\boldsymbol{\omega}_{1/0} = \frac{d_{<0>}}{dt}\,\boldsymbol{\omega}_{1/0} \tag{2.8}$$

$$\boldsymbol{\omega}_{1/0} = -\boldsymbol{\omega}_{0/1} \tag{2.9}$$

$$\boldsymbol{\omega}_{j/j} = 0 \tag{2.10}$$

$$\boldsymbol{\omega}_{c/a} = \boldsymbol{\omega}_{c/b} + \boldsymbol{\omega}_{b/a} \tag{2.11}$$

where the last one follows from

$$\frac{d_{<a>}}{dt}\,\boldsymbol{\rho} = \frac{d_{<b>}}{dt}\,\boldsymbol{\rho} + \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{\rho} \tag{2.12}$$

$$\frac{d_{<b>}}{dt}\,\boldsymbol{\rho} = \frac{d_{<c>}}{dt}\,\boldsymbol{\rho} + \boldsymbol{\omega}_{c/b} \wedge \boldsymbol{\rho} \tag{2.13}$$

$$\frac{d_{<a>}}{dt}\,\boldsymbol{\rho} = \frac{d_{<c>}}{dt}\,\boldsymbol{\rho} + \boldsymbol{\omega}_{c/a} \wedge \boldsymbol{\rho} \tag{2.14}$$

the substitution of (2.13) and (2.14) in (2.12).

Velocity composition rules Equations (2.7) and (2.11) can be used to calculate the relationship among the linear and angular velocities of a chain of n reference frames. From equation (2.11) follows

$$\boldsymbol{\omega}_{n/0} = \sum_{i=1}^{n} \boldsymbol{\omega}_{i/i-1} \tag{2.15}$$

As far as the linear velocity is concerned the linear velocity vector $\mathbf{v}_{i/j}$ of frame $< i >$ with respect to frame $< j >$ is defined as:

$$\mathbf{v}_{i/j} \triangleq \frac{d_{<j>}}{dt}\,(O_i - O_j) \tag{2.16}$$

so that

$$\begin{aligned}
\mathbf{v}_{i/0} &= \frac{d_{<0>}}{dt}\left(O_i - O_0\right) = \frac{d_{<0>}}{dt}\left(O_i - O_{i-1}\right) + \frac{d_{<0>}}{dt}\left(O_{i-1} - O_0\right) = \\
&= \frac{d_{<i-1>}}{dt}\left(O_i - O_{i-1}\right) + \boldsymbol{\omega}_{i-1/0} \wedge \left(O_i - O_{i-1}\right) + \mathbf{v}_{i-1/0} \qquad \Rightarrow \\
\mathbf{v}_{i/0} &= \mathbf{v}_{i/i-1} + \boldsymbol{\omega}_{i-1/0} \wedge \mathbf{r}_{i-1,i} + \mathbf{v}_{i-1/0} \qquad (2.17)
\end{aligned}$$

From equation (2.17) follows:

$$\begin{aligned}
\mathbf{v}_{n/0} + \sum_{i=1}^{n} \mathbf{v}_{i/0} &= \sum_{i=1}^{n} \mathbf{v}_{i/i-1} + \sum_{i=1}^{n} (\boldsymbol{\omega}_{i-1/0} \wedge \mathbf{r}_{i-1,i}) + \sum_{i=1}^{n} \mathbf{v}_{i-1/0} + \mathbf{v}_{n/0} \Rightarrow \\
\mathbf{v}_{n/0} &= \sum_{i=1}^{n} \mathbf{v}_{i/i-1} + \sum_{i=1}^{n} (\boldsymbol{\omega}_{i-1/0} \wedge \mathbf{r}_{i-1,i}) \qquad (2.18)
\end{aligned}$$

as $\sum_{i=1}^{n} \mathbf{v}_{i/0} = \sum_{i=1}^{n} \mathbf{v}_{i-1/0} + \mathbf{v}_{n/0}$. To understand the nature of the second sum on the right hand side of equation (2.18) notice that

$$\begin{aligned}
&\sum_{i=1}^{n} (\boldsymbol{\omega}_{i-1/0} \wedge \mathbf{r}_{i-1,i}) = \boldsymbol{\omega}_{0/0} \wedge \mathbf{r}_{0,1} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,2} + \boldsymbol{\omega}_{2/0} \wedge \mathbf{r}_{2,3} + \boldsymbol{\omega}_{3/0} \wedge \mathbf{r}_{3,4} + \cdots = \\
= \;& \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,2} + \left[(\boldsymbol{\omega}_{2/1} + \boldsymbol{\omega}_{1/0}) \wedge \mathbf{r}_{2,3}\right] + \left[(\boldsymbol{\omega}_{3/2} + \boldsymbol{\omega}_{2/1} + \boldsymbol{\omega}_{1/0}) \wedge \mathbf{r}_{3,4}\right] + \cdots = \\
= \;& \boldsymbol{\omega}_{1/0} \wedge (\mathbf{r}_{1,2} + \mathbf{r}_{2,3} + \mathbf{r}_{3,4} + \cdots) + \boldsymbol{\omega}_{2/1} \wedge (\mathbf{r}_{2,3} + \mathbf{r}_{3,4} + \mathbf{r}_{4,5} + \cdots) + \cdots = \\
= \;& \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,n} + \boldsymbol{\omega}_{2/1} \wedge \mathbf{r}_{2,n} + \boldsymbol{\omega}_{3/2} \wedge \mathbf{r}_{3,n} + \cdots \qquad \Rightarrow \\
&\sum_{i=1}^{n} (\boldsymbol{\omega}_{i-1/0} \wedge \mathbf{r}_{i-1,i}) = \sum_{i=1}^{n} (\boldsymbol{\omega}_{i/i-1} \wedge \mathbf{r}_{i,n}) \qquad (2.19)
\end{aligned}$$

Replacing equation (2.19) in (2.18) the linear velocity of the n-th frame of a kinematic chain is calculated as a function of the relative velocities of each other frame with respect to the previous one, i.e.:

$$\mathbf{v}_{n/0} = \sum_{i=1}^{n} (\mathbf{v}_{i/i-1} + \boldsymbol{\omega}_{i/i-1} \wedge \mathbf{r}_{i,n}) \qquad (2.20)$$

Considering the special case $n = 2$ both, the rigid body velocity composition rule, and the Galilean velocity composition rule can be deduced from (2.20). By direct calculation

$$\mathbf{v}_{2/0} = \mathbf{v}_{1/0} + \mathbf{v}_{2/1} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,2}$$

so that if the origin of the third frame is called p instead then O_2 follows

$$\mathbf{v}_{p/0} = \mathbf{v}_{1/0} + \mathbf{v}_{p/1} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,p} \qquad (2.21)$$

with obvious meaning of notation. The relative velocity $\mathbf{v}_{p/1}$of point p with respect to $O_1$is null if p and the frame $< 1 >$ are fixed to the same rigid body. Thus if $< 0 >$ is a fixed (*absolute*) reference and $< 1 >$ moves attached to a rigid body (*relative* reference), each point p of the rigid body will have absolute velocity

$$\mathbf{v}_{p/0} = \mathbf{v}_{1/0} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,p} \tag{2.22}$$

being $\mathbf{r}_{p,1} = p - O_1$ the position vector of p respect to O_1 by definition. As far as the Galilean velocity composition rule is concerned, equation (2.21) can be written as

$$\begin{aligned}
\mathbf{v}_{p/0} - \mathbf{v}_{1/0} &= \frac{d_{<0>}}{dt}\,(p - O_0 - O_1 + O_0) = \\
&= \frac{d_{<0>}}{dt}\,\mathbf{r}_{1,p} = \mathbf{v}_{p/1} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,p} \qquad \Rightarrow \\
\frac{d_{<0>}}{dt}\,\mathbf{r}_{1,p} &= \frac{d_{<1>}}{dt}\,\mathbf{r}_{1,p} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{1,p} \qquad (2.23)
\end{aligned}$$

which is the desired Galilean velocity composition rule equation.

2.1.3 On useful vector operations properties

As n-dimensional vector quantities are assumed to be elements of $\Re^{n\times 1}$ the scalar product operation introduced in (2.3) with the symbol $\cdot$ can be also thought of as a row by column product, i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\ \mathbf{b}\ \forall\ \mathbf{a}, \mathbf{b} \in \Re^{n\times 1}$. The vector product $\mathbf{a} \wedge \mathbf{b}$ can be thought of as

$$\mathbf{a} \wedge \mathbf{b} \triangleq [\mathbf{a}\wedge]\ \mathbf{b} \triangleq S(\mathbf{a})\ \mathbf{b} \triangleq \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2.24}$$

and more generally any skew-symmetric operator can be thought of as a vector product. This is an important property that may be worthwhile showing. Consider a generic skew-symmetric operator A: by definition of skew-symmetry given any vectors $\mathbf{u}$,$\mathbf{v}$ the following must hold $A(\mathbf{u}) \cdot \mathbf{v} = -\mathbf{u} \cdot A(\mathbf{v})$ which is equivalent to the statement that for any skew-symmetric operator A and any vector $\mathbf{v}$, $A(\mathbf{v}) \cdot \mathbf{v} = 0$. Given a generic skew-symmetric operator A and an othonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, consider the vector $\mathbf{a} \triangleq \frac{1}{2}\sum_{i=1}^{3} \mathbf{e}_i \wedge A(\mathbf{e}_i)$ (*axis vector*) and the for any vector $\mathbf{v}$ the following holds:

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{v} &= \left(\frac{1}{2}\sum_{i=1}^{3} \mathbf{e}_i \wedge A(\mathbf{e}_i)\right) \wedge \mathbf{v} = \frac{1}{2}\sum_{i=1}^{3} \left((\mathbf{v} \cdot \mathbf{e}_i)\ A(\mathbf{e}_i) - (A(\mathbf{e}_i) \cdot \mathbf{v})\ \mathbf{e}_i\right) = \\
&= \frac{1}{2}\sum_{i=1}^{3} \left(v_i\ A(\mathbf{e}_i) + (A(\mathbf{v}) \cdot \mathbf{e}_i)\ \mathbf{e}_i\right) = A(\mathbf{v})
\end{aligned}$$

being $-A(\mathbf{e}_i) \cdot \mathbf{v} = A(\mathbf{v}) \cdot \mathbf{e}_i$ by definition of skew-symmetry of A and $\sum_{i=1}^{3}(\mathbf{v} \cdot \mathbf{e}_i)\, A(\mathbf{e}_i) = \sum_{i=1}^{3} v_i\, A(\mathbf{e}_i) = \sum_{i=1}^{3} A(v_i\mathbf{e}_i) = A(\mathbf{v})$ by the linearity of A. Two simple consequences of the above result are

- $\mathbf{a}$ is unique (suppose $\mathbf{a}$, $\mathbf{b}$ such that $A(\mathbf{v}) = \mathbf{a} \wedge \mathbf{v} = \mathbf{b} \wedge \mathbf{v}$ then $(\mathbf{a} - \mathbf{b}) \wedge \mathbf{v} = 0 \Rightarrow \mathbf{a} = \mathbf{b}$).
- A has only one real eigenvalue $\lambda = 0$ relative to the eigenvector $\mathbf{a}$.

A frequent kind of vector operation in kinematic and dynamic calculations is the double vector product (2.3) $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ which is linear in each of the three vectors. Noting by direct calculation that for any three vectors $(\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} = [\mathbf{c}\,\mathbf{a}^T]\, \mathbf{b}$ where

$$[\mathbf{c}\,\mathbf{a}^T] = \begin{pmatrix} c_1 a_1 & c_1 a_2 & c_1 a_3 \\ c_2 a_1 & c_2 a_2 & c_2 a_3 \\ c_3 a_1 & c_3 a_2 & c_3 a_3 \end{pmatrix} \tag{2.25}$$

$[\mathbf{c}\,\mathbf{a}^T]$ is the *external vector product*, the double vector product (2.3) can be written as

$$\begin{aligned} \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) = \\ &= ([\mathbf{b}\,\mathbf{c}^T - \mathbf{c}\,\mathbf{b}^T])\,\mathbf{a} && (2.26) \\ &= (I_{3\times3}(\mathbf{a} \cdot \mathbf{c}) - [\mathbf{c}\,\mathbf{a}^T])\,\mathbf{b} && (2.27) \\ &= ([\mathbf{b}\,\mathbf{a}^T] - I_{3\times3}(\mathbf{a} \cdot \mathbf{b}))\,\mathbf{c} && (2.28) \end{aligned}$$

being $I_{3\times3}$ the 3×3 identical matrix.

Another useful result in vector analysis is *Helmholtz's theorem*: any finite, uniform, continuous and vanishing at infinity vector field $\mathbf{F}$ may be written as the sum of the gradient of a scalar φ and the curl of a zero divergence vector $\mathbf{a}$, i.e.[33]

$$\begin{aligned} \forall\, \mathbf{F} &\in \Re^{3\times1} \text{ uniform, finite and vanashing at infinite} \Rightarrow \\ \exists\, \varphi &\in \Re, \mathbf{a} \in \Re^{3\times1} \mid \mathbf{F} = \nabla\varphi + \nabla \wedge \mathbf{a}\ ,\ \nabla \cdot \mathbf{a} = 0 \end{aligned}$$

As the divergence of the rotor of any vector is identically null, from the above follows that the divergence of a vector field $\mathbf{F}$ satisfying the above hypothesis can be written as the laplacian of a scalar, i.e.

$$\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla\varphi) + \nabla \cdot (\nabla \wedge \mathbf{a}) = \nabla^2\varphi$$

moreover if the rotor of $\mathbf{F}$ is null ($\mathbf{F}$ is *conservative*) then $\mathbf{a}$ itself is identically null [33]. These results are useful in the hydrodynamic theory of ideal fluids that will be discussed in the next chapter.