# Chapter 3
# Dynamics

Within this chapter the dynamics of a robotic structure will be revised and extended to a fluid environment. The Euler-Newton formulation will be adopted.

## 3.1 Rigid body Newton-Euler equations

With reference to figure (3.1) the Newton-Euler equations of motion of the rigid body will be outlined. Reference $< 0 >$ having origin in $o$ is inertial, while reference $< 1 >$ having origin in $u$ is fixed to the rigid body having center of mass in point $c$. Indicating with $\rho(\mathbf{r})$ the density of the body, with $m$ it's mass, with $V$ it's volume, with $\mathbf{r}_{u,c} \triangleq (c-u)$ (Grassman's notation, see Chapter 2) the position of $c$ with respect to $u$ and with $T$ it's kinetic energy the following hold by definition

$$m \triangleq \int_V \rho(\mathbf{r}_{u,p})\, dV \tag{3.1}$$

$$m\, \mathbf{r}_{u,c} \triangleq \int_V \rho(\mathbf{r}_{u,p})\mathbf{r}_{u,p}\, dV \tag{3.2}$$

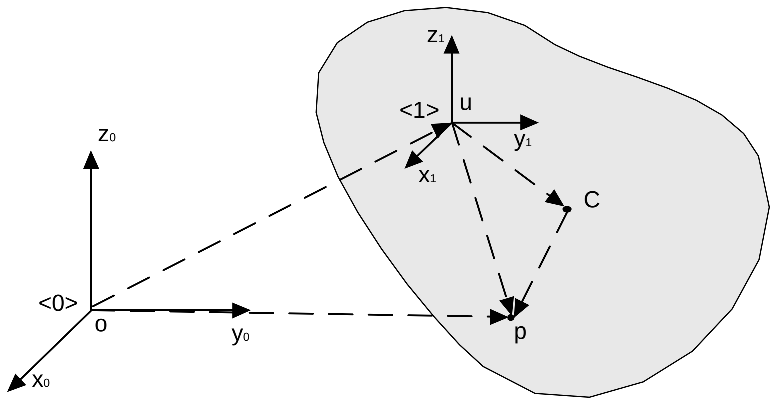
$$T \triangleq \frac{1}{2}\int_V \rho(\mathbf{r}_{o,p})\, \mathbf{v}_{p/o}\cdot\mathbf{v}_{p/o}\, dV \tag{3.3}$$

being $p$ a generic point of the rigid body, $dV$ an infinitesimal volume element equal to $d^3(p-u)$ in (3.1) and (3.2) and to $d^3(p-o)$ in (3.3). According to equation (2.17) the velocity $\mathbf{v}_{p/o}$ of equation (3.3) can be written as

$$\begin{aligned}
\mathbf{v}_{p/o} &= \frac{d_{<0>}}{dt}(p-o) = \frac{d_{<0>}}{dt}(p-c) + \frac{d_{<0>}}{dt}(c-o) = \\
&= \mathbf{v}_{p/c} + \boldsymbol{\omega}_{c/o}\wedge\mathbf{r}_{c,p} + \mathbf{v}_{c/o} \Rightarrow \\
\mathbf{v}_{p/o} &= \mathbf{v}_{c/o} + \boldsymbol{\omega}_{c/o}\wedge\mathbf{r}_{c,p}
\end{aligned} \tag{3.4}$$

being $\mathbf{v}_{p/c} = \frac{d_{<1>}}{dt}(p-c) = 0\ \forall\, p$ by definition of rigid body. Replacing equation (3.4) in (3.3) Köenig's theorem is derived:

$$\begin{aligned}
T &= \frac{1}{2}\int_V \rho\, \mathbf{v}_{p/o}\cdot\mathbf{v}_{p/o}\, dV = \frac{1}{2}\int_V \rho\, \mathbf{v}_{c/o}\cdot\mathbf{v}_{c/o}\, dV + \\
&+ \frac{1}{2}\int_V \rho\, (\boldsymbol{\omega}_{c/o}\wedge\mathbf{r}_{c,p})\cdot(\boldsymbol{\omega}_{c/o}\wedge\mathbf{r}_{c,p})\, dV = \\
&= \frac{1}{2}v_{c/o}^2\int_V \rho\, dV + \frac{1}{2}\boldsymbol{\omega}_{c/o}\cdot\int_V \rho\, \mathbf{r}_{c,p}\wedge(\boldsymbol{\omega}_{c/o}\wedge\mathbf{r}_{c,p})\, dV \Rightarrow
\end{aligned}$$




3.1.Rigid body, refer to text

$$T = \frac{1}{2} m v_{c/o}^2 + \frac{1}{2} \boldsymbol{\omega}_{c/o} \cdot I_c \boldsymbol{\omega}_{c/o} \tag{3.5}$$

where the inertia matrix operator $I_c$ with respect to the center of mass has been introduced. By definition of inertia operator and remembering equations (2.26), (2.27) and (2.28) the inertia operator with respect an arbitrary point, e.g. $u$, is

$$I_u \boldsymbol{\omega} \triangleq \int_V \rho \, \mathbf{r}_{u,p} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}_{u,p}) \, dV = \tag{3.6}$$

$$= \left( \int_V \rho \, [I_{3\times 3} (\mathbf{r}_{u,p} \cdot \mathbf{r}_{u,p}) - \mathbf{r}_{u,p} \mathbf{r}_{u,p}^T] \, dV \right) \boldsymbol{\omega} \Rightarrow$$

$$I_u \triangleq \int_V \rho \, [I_{3\times 3} (\mathbf{r}_{u,p} \cdot \mathbf{r}_{u,p}) - \mathbf{r}_{u,p} \mathbf{r}_{u,p}^T] \, dV. \tag{3.7}$$

The inertia tensor thus defined is symmetric and positive definite. With reference to figure (3.1) notice that replacing $\mathbf{r}_{u,p} = \mathbf{r}_{c,p} - \mathbf{r}_{c,u}$ in equation (3.7) the *parallel axis theorem* is immediately derived, i.e.,

$$I_u = \int_V \rho(\mathbf{r}_{u,p}) \, [I_{3\times 3} (\mathbf{r}_{c,p} \cdot \mathbf{r}_{c,p}) - \mathbf{r}_{c,p} \mathbf{r}_{c,p}^T] \, dV +$$

$$+ \int_V \rho(\mathbf{r}_{u,p}) \, [I_{3\times 3} (\mathbf{r}_{c,u} \cdot \mathbf{r}_{c,u}) - \mathbf{r}_{c,u} \mathbf{r}_{c,u}^T] \, dV +$$

$$- \int_V \rho(\mathbf{r}_{u,p}) \, [I_{3\times3}(2 \, \mathbf{r}_{c,p} \cdot \mathbf{r}_{c,u}) - \mathbf{r}_{c,p}\mathbf{r}_{c,u}^T - \mathbf{r}_{c,u}\mathbf{r}_{c,p}^T] \, dV.$$

The first integral is just $I_c$, by definition, the second one is equal to $m \, [I_{3\times3}(\mathbf{r}_{c,u} \cdot \mathbf{r}_{c,u}) - \mathbf{r}_{c,u}\mathbf{r}_{c,u}^T]$ being $\mathbf{r}_{u,c}$ constant, and the third one is null as by definition of center of mass $c$ the following holds: $m \, \mathbf{r}_{c,c} \triangleq \int_V \rho(\mathbf{r}_{u,p}) \, \mathbf{r}_{c,p} \, dV = 0$. Thus for any point $u$ the parallel axis theorem states that

$$I_u = I_c + m \, [I_{3\times3}(\mathbf{r}_{c,u} \cdot \mathbf{r}_{c,u}) - \mathbf{r}_{c,u}\mathbf{r}_{c,u}^T] \tag{3.8}$$

As the hydrodynamic forces applied on a body are usually derived with respect to the local reference frame, the standard Newton equations of a rigid body will be now calculated in reference $< 1 >$. With reference to equation (2.7) the absolute velocity of a generic point $p$ of the rigid body in figure (3.1) is

$$\begin{aligned}
\mathbf{v}_{p/o} &= \frac{d_{<0>}}{dt}(p - o) = \frac{d_{<0>}}{dt}(u - o) + \frac{d_{<0>}}{dt}(p - u) = \\
&= \mathbf{v}_{u/o} + \frac{d_{<1>}}{dt}(p - u) + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{u,p} \Rightarrow \\
\mathbf{v}_{p/o} &= \mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{u,p}
\end{aligned} \tag{3.9}$$

being $\frac{d_{<1>}}{dt}(p - u) = \frac{d_{<1>}}{dt}\mathbf{r}_{u,p} = 0$ by definition of rigid body. Notice that all involved vectors are free vectors although according to equation (2.7) the most "natural" reference frame where to project $\mathbf{v}_{u/o}$, $\boldsymbol{\omega}_{1/0}$ and $\mathbf{r}_{u,p}$ is the local reference $< 1 >$. The absolute acceleration will be:

$$\begin{aligned}
\mathbf{a}_{p/o} &= \frac{d_{<0>}}{dt}\mathbf{v}_{p/o} = \frac{d_{<1>}}{dt}\mathbf{v}_{p/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{v}_{p/o} = \\
&= \frac{d_{<1>}}{dt}(\mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{u,p}) + \boldsymbol{\omega}_{1/0} \wedge (\mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{u,p}) \Rightarrow \\
\mathbf{a}_{p/o} &= \frac{d_{<1>}}{dt}\mathbf{v}_{u/o} + \left(\frac{d_{<1>}}{dt}\boldsymbol{\omega}_{1/0}\right) \wedge \mathbf{r}_{u,p} + \\
&\quad + \boldsymbol{\omega}_{1/0} \wedge \mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge (\boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{u,p})
\end{aligned} \tag{3.10}$$

This equation can be used to calculate the Newton force equation

$$\frac{d_{<0>}}{dt} \int_V \rho(\mathbf{r}_{u,p})\mathbf{v}_{p/o} dV = \sum_i \mathbf{F}_i^{external}$$

for a rigid body having a time invariant density, in the local reference. By direct calculation it follows that

$$m\left(\frac{d_{<1>}}{dt}\mathbf{v}_{u/o}+\left(\frac{d_{<1>}}{dt}\boldsymbol{\omega}_{1/0}\right)\wedge\mathbf{r}_{u,c}+\boldsymbol{\omega}_{1/0}\wedge\mathbf{v}_{u/o}+\boldsymbol{\omega}_{1/0}\wedge(\boldsymbol{\omega}_{1/0}\wedge\mathbf{r}_{u,c})\right)$$
$$=\sum_i\mathbf{F}_i^{external} \tag{3.11}$$

where $m$ is the total time-constant mass of the body, $r_{u,c}$ its center of mass position relative to reference $<1>$ as given by equation (3.2), $\mathbf{v}_{u/o}$ is the absolute velocity of references $<1>$ origin $u$, $\boldsymbol{\omega}_{1/0}$ its absolute angular velocity and $\sum_i\mathbf{F}_i^{external}$ is the sum of all external forces applied on the body. Again notice that by construction the most natural reference frame where to project all the free vectors in equation (3.11) is $<1>$. In particular adopting the standard SNAME notation for marine systems the following hold:

$$\begin{aligned}
{}^1(\mathbf{v}_{u/o}) &= (u,v,w)^T\\
{}^1(\boldsymbol{\omega}_{1/0}) &= (p,q,r)^T\\
{}^1\left(\frac{d_{<1>}}{dt}\mathbf{v}_{u/o}\right) &= (du/dt,dv/dt,dw/dt)^T\\
{}^1\left(\frac{d_{<1>}}{dt}\boldsymbol{\omega}_{1/0}\right) &= (dp/dt,dq/dt,dr/dt)^T\\
u &= surge\\
v &= sway\\
w &= heave\\
p &= roll\\
q &= pitch\\
r &= yaw
\end{aligned}$$

The Newton equation for the rotational dynamics is related to the absolute angular momentum balance. In particular calling $\mathbf{N}_u$ the force moment about point $u$ by definition the following holds:

$$\begin{aligned}
\sum_i\mathbf{N}_{u,i}^{external} &= \int_V(\mathbf{r}_{u,p}\wedge\frac{d_{<0>}}{dt}\mathbf{v}_{p/o})\rho(\mathbf{r}_{u,p})\,dV=\\
&= \int_V(\mathbf{r}_{u,p}\wedge\mathbf{a}_{p/o})\rho(\mathbf{r}_{u,p})\,dV
\end{aligned}\tag{3.12}$$

Substituting equation (3.10) in (3.12):

$$\int_V(\mathbf{r}_{u,p}\wedge\mathbf{a}_{p/o})\rho(\mathbf{r}_{u,p})\,dV=$$

$$\begin{aligned} &= \int_V \left( \mathbf{r}_{u,p} \wedge \left( \frac{d_{<1>}}{dt} \mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{v}_{u/o} \right) \right) \rho(\mathbf{r}_{u,p})\, dV + \\ &+ \int_V \left( \mathbf{r}_{u,p} \wedge \left( \left( \frac{d_{<1>}}{dt} \boldsymbol{\omega}_{1/0} \right) \wedge \mathbf{r}_{u,p} \right) \right) \rho(\mathbf{r}_{u,p})\, dV + \\ &+ \int_V (\mathbf{r}_{u,p} \wedge (\boldsymbol{\omega}_{1/0} \wedge (\boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{u,p}))) \rho(\mathbf{r}_{u,p})\, dV = \sum_i \mathbf{N}_{u,i}^{external} \end{aligned}$$

where the integral in the second line is $m\ \mathbf{r}_{u,c} \wedge \left( \frac{d_{<1>}}{dt} \mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{v}_{u/o} \right)$, the one in the third is $I_u \left( \frac{d_{<1>}}{dt} \boldsymbol{\omega}_{1/0} \right)$by equation (3.6), and the on in the fourth can be shown to be $\boldsymbol{\omega}_{1/0} \wedge I_u \boldsymbol{\omega}_{1/0}$ by some vector manipulations based on the properties shown in paragraph 2.1.3. The second Newton equation for a rigid body can be thus expressed in the local reference frame having origin in $u$, as

$$\begin{aligned} & I_u \left( \frac{d_{<1>}}{dt} \boldsymbol{\omega}_{1/0} \right) + \boldsymbol{\omega}_{1/0} \wedge I_u \boldsymbol{\omega}_{1/0} + m\ \mathbf{r}_{u,c} \wedge \left( \frac{d_{<1>}}{dt} \mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{v}_{u/o} \right) \\ = & \sum_i \mathbf{N}_{u,i}^{external} \end{aligned} \tag{3.13}$$

Equations (3.11) and (3.13) can be written in matrix (space notation) form as:

$$M\ \frac{d_{<1>}}{dt} \boldsymbol{\nu} + C(\boldsymbol{\omega}_{1/0})\ \boldsymbol{\nu} = \boldsymbol{\tau}^{ext} \tag{3.14}$$

$$\boldsymbol{\nu} = (\mathbf{v}_{u/o}^T, \boldsymbol{\omega}_{1/0}^T)^T \in \Re^{6 \times 1} \tag{3.15}$$

being $\boldsymbol{\nu}$ the generalized velocity, $M \in \Re^{6\times 6}$ the inertia operator, $C(\boldsymbol{\omega}) \in \Re^{6\times 6}$ the Coriolis and centripetal operator and $\boldsymbol{\tau}^{ext} = \sum_i (\mathbf{F}_i^T, \mathbf{N}_{u,i}^T)^{T\ external} \in \Re^{6\times 1}$ the generalized torque applied to the body. By direct calculation it can be shown that the inertia and Coriolis-centripetal operators are given by

$$M \triangleq \begin{pmatrix} m\ I_{3\times 3} & -m\ S(\mathbf{r}_{u,c}) \\ m\ S(\mathbf{r}_{u,c}) & I_u \end{pmatrix} \tag{3.16}$$

$$C(\boldsymbol{\omega}_{1/0}) \triangleq \begin{pmatrix} m\ S(\boldsymbol{\omega}_{1/0}) & -m\ S(\boldsymbol{\omega}_{1/0})\ S(\mathbf{r}_{u,c}) \\ m\ S(\mathbf{r}_{u,c})\ S(\boldsymbol{\omega}_{1/0}) & -S(I_u \boldsymbol{\omega}_{1/0}) \end{pmatrix} \tag{3.17}$$

being $S$ the skew symmetric vector product operator defined by equation (2.24). It can be shown [32] that while the parametrization of the positive definite rigid body inertia matrix given in equation (3.16) is unique, the Coriolis-centripetal matrix can be parametrized in a non-unique skew symmetric form. The one given in equation (3.17) has the advantage of depending only on $\boldsymbol{\omega}_{1/0}$, but as shown in [32] other skew symmetric parametrizations depending on $\boldsymbol{\nu} = (\mathbf{v}_{u/o}^T, \boldsymbol{\omega}_{1/0}^T)^T$ are possible. To characterize the dynamics of a rigid body in a fluid environment the right hand side $\boldsymbol{\tau}^{ext}$ of equation (3.14) has to be calculated explicitly.

## 3.2 Fluid forces and moments on a rigid body

When a body moves in a fluid environment it experiences external forces due to the interaction between itself and the fluid. As can be imagined even intuitively, all these forces are somehow proportional to the fluids density and to the relative speed and acceleration between the body and the fluid. When a body moves in atmospheric air at low speeds, as for the majority of the robotic applications, these forces are negligeable. On the contrary in underwater applications, due to the high density of water, these forces are never negligeable even at the lowest speeds. The calculation of hydrodynamic generalized forces on a rigid body is a classical and well known topic in fluid dynamics theory that will thus be here only revised in view of the robotic applications of interest. For a more detailed discussion refer to [34] [35] [36].

### 3.2.1 The Navier Stokes equation

The Navier Stokes equation is the equation of motion of an infinitesimal volume of newtonian, incompressible and time-constant density fluid. To derive this equation the following notation will be used: $\rho$ will denote the fluid density (dimensions $[Kg/m^3]$), $\mathbf{F}$ the force per unit volume (dimensions $[N/m^3]$), $\boldsymbol{\tau}_{ij}$ the stress tensor (dimensions $[N/m^2]$), $p$ the pressure (dimensions $[N/m^2]$), $V$ a volume element (dimensions $[m^3]$) of surface $S$ (dimensions $[m^2]$) having unit normal vector $\mathbf{n} = (n_1, n_2, n_3)^T$(dimensions $[m]$), $\mathbf{u} = (u_1, u_2, u_3)^T$ the fluids local velocity (dimensions $[m/s]$) with respect to an inertial frame. A preliminary result for the derivation of the Navier Stokes equation and other important fluid dynamic properties is the *transport theorem*. Given a differentiable function $f(\mathbf{x}, t)$, the quantity $I(t) = \int\int\int_{V(t)} f(\mathbf{x}, t) dV$ where $V(t)$ is a time evolving volume of surface $S(t)$ has time derivative

$$\frac{d}{dt} I(t) = \int\int\int_{V(t)} \frac{\partial}{\partial t} f(\mathbf{x}, t)\, dV + \int\int_{S(t)} f(\mathbf{x}, t) U_n \, dS \tag{3.18}$$

being $U_n$ the normal velocity of $S$. An important special case of equation (3.18) is related to the situation where the volume $V(t)$ and the surface $S(t)$ are relative to the same fluid particles. In such situation $U_n \triangleq \mathbf{u}^T \mathbf{n} = u_i n_i$ (where repeated indexes are to be interpreted as summed) and by applying the Gauss theorem to the second integral on the right hand side of (3.18) the following holds

$$\frac{d}{dt} \int\int\int_{V(t)} f(\mathbf{x}, t)\, dV = \int\int\int_{V(t)} \left( \frac{\partial}{\partial t} f(\mathbf{x}, t) + \nabla \cdot f(\mathbf{x}, t) \mathbf{u} \right) dV \tag{3.19}$$

being $\nabla \triangleq (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^T$ the gradient operator. Substituting the fluid density $\rho$ to the function $f$ for a generic volume element in equation (3.19) the principle of mass conservation implies $\int\int\int_{V(t)} \left(\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla \cdot \rho(\mathbf{x},t)\mathbf{u}\right) dV = 0$ and thus

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla \cdot \rho(\mathbf{x},t)\mathbf{u} = 0 \tag{3.20}$$

for the arbitrary of $V$. If the fluid is assumed incompressible and of constant density in time it follows

$$\left.\begin{array}{l} \frac{\partial}{\partial t}\rho(\mathbf{x},t) = 0 \\ \nabla\rho(\mathbf{x},t) = 0 \end{array}\right\} \Rightarrow \nabla \cdot \mathbf{u} = \frac{\partial}{\partial x_j} u_j = 0 \tag{3.21}$$

Equation (3.19) can be applied to the momentum $\rho u_i$ conservation of a generic volume $V$ of fluid yielding

$$\begin{aligned} \frac{d}{dt}\int\int\int_{V(t)} \rho u_i \, dV &= \int\int\int_{V(t)} \left(\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j)\right) dV = \\ &= \int\int\int_{V(t)} \left(\frac{\partial}{\partial x_j}\tau_{ij} + F_i\right) dV \end{aligned}$$

As the choice of $V$ is arbitrary this last equation implies

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial}{\partial x_j}\tau_{ij} + F_i$$

which in the hypothesis stated in (3.21) implies the Euler's equation of an incompressible time-constant density fluid

$$\frac{\partial}{\partial t}u_i + u_j \frac{\partial}{\partial x_j}u_i = \frac{1}{\rho}\left(\frac{\partial}{\partial x_j}\tau_{ij} + F_i\right) \tag{3.22}$$

To finally derive the Navier Stokes equation some further hypothesis on the stress tensor $\boldsymbol{\tau}$ must be made. The balance of moments acting on a free element of fluid $dV$ implies its symmetry, i.e., $\boldsymbol{\tau}_{ij} = \boldsymbol{\tau}_{ji} \; \forall \, ij$. Moreover it can be shown [34] that the most general form of stress tensor of an isotropic fluid satisfying (3.21) and whose volume element $dV$ does not undergo deformation when moving as a rigid body, i.e. with a velocity $\mathbf{v} + \boldsymbol{\omega} \wedge \mathbf{r}$ being $\mathbf{v}$ and $\boldsymbol{\omega}$ constant, is

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right) \forall\, i \neq j \tag{3.23}$$

being $\delta_{ij}$ the Kronecker delta symbol, $p$ the pressure and $\mu$ the *viscous shear coefficient* (dimensions $[Kg/ms]$). Equation (3.23) defines a *newtonian* fluid; notice that the vast majority of fluids, including air and water, indeed exhibit a newtonian behaviour. Replacing equation (3.23) in (3.22) and using property (3.21) the Navier Stokes equation is derived

$$\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\,\nabla^2\mathbf{u} + \frac{1}{\rho}\mathbf{F} \tag{3.24}$$

being $\nu$ the *kinematic viscosity* $\nu = \mu/\rho$ (dimensions $[m^2/s]$). Equations (3.21) and

(3.24) with suitable boundary conditions totally describe the flow of a newtonian incompressible time-constant density fluid, but are of relevant practical use only in those very special cases where the geometry and boundary conditions of the problem allow to find an analytical solution, e.g. $2D$ flows past two parallel walls (*Couette flow*) or flow in a cylindrical pipe (*Poiseuille flow*). Fortunately it can be shown theoretically and experimentally that the viscous effects in a fluid flow are relevant only in a very limited fluid volume next to the separating surface with a rigid body (*thin boundary layer theory*) and that they decay very rapidly in the bulk of a fluid. As a consequence the standard approach to estimate fluid forces on a rigid body consists in calculating all inertial pressure effects as if the fluid was inviscid, i.e. $\nu = 0$, and then to add the viscous effects estimated by the thin boundary layer theory or experimentally.

### 3.2.2 Viscous effects

To get a qualitative understanding of viscous effects in a fluid flow it may be useful to calculate the order of magnitude of the ratio between inertial and viscous forces in a general fluid dynamic problem. Assuming that the problem is characterized by velocity $U$, length $l$, viscous shear coefficient $\mu$, gravitational acceleration $g = 9.81\ m/s^2$, and fluid density $\rho$, consider the ratios [34]

$$F^{1/2} \triangleq \frac{Inertial\ force}{Gravitational\ force} = \frac{\rho U^2 l^2}{\rho g l^3} = U^2/gl \tag{3.25}$$

$$R \triangleq \frac{Inertial\ force}{Viscous\ force} = \frac{\rho U^2 l^2}{\mu U l} = \rho U l/\mu = Ul/\nu \tag{3.26}$$

being the first the square root of the *Froude Number* and the second the *Reynolds Number* of the specific problem. As both fresh and salt water have a kinematic viscosity $\nu$ ranging form $0.8 \cdot 10^{-6}\ m^2/s$ to $1.8 \cdot 10^{-6}\ m^2/s$ for temperatures between $0^\circ$ and $30^\circ$ degrees Celsius, it follows that the Reynolds number for typical underwater robotic systems of $1m$ length-scale and $1m/s$ velocity-scale is $R \in [0.6, 1.2] \cdot 10^6$. This value actually suggests that in the bulk of the fluid viscous effects may be neglected with respect to the inertial ones. Notice that the different scaling properties of the Reynolds Number and the Froude Number with respect to variables of interest $U$, $l$ and $\nu$ are at the basis of the difficulty in simulating the behaviour of large marine systems by scaled models. Roughly speaking viscous forces on a rigid body can be thought of as *drag* forces and lift *forces*. The former are parallel to the relative velocity of the body with respect to the fluid and the latter are normal to it.

#### 3.2.2.1 Viscous drag forces

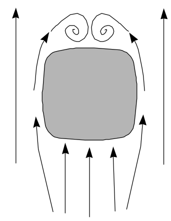
By dimensional analysis it can be argued [34] that the drag force $F_{drag}$ experienced by a sphere of diameter $d$ moving in a fluid of density $\rho$ with velocity $U$ can be written as $F_{drag} = \frac{1}{2}\rho U^2 S\ C_d(R)$ being $S = \pi d^2/4$ the frontal area of the sphere and $C_d(R)$ the

Reynolds dependent drag coefficient. Experimental data reported in [34] relative to a wide range of different sphere diameters $d$ and different fluids shows the validity of that equation. Moreover the plot of $C_d$ versus $R$ shows a sharp discontinuity at about $R = 3 \cdot 10^5$ relative to the transition between the so called *laminar* and *turbulent* regimes. For more general slender body geometries it is assumed that the drag coefficient $C_d$ can be thought of as the sum of a *frictional* term $C_f$ and a *pressure* or *form* term $C_p$, i.e. $C_d = C_f + C_p$. The frictional term is due to the shear drag experienced by the surface of the body travelling parallel to the relative velocity $U$, while the pressure term is due to the frontal surface of the body normal to $U$. The frictional drag coefficient $C_f$ on a slender body is usually modeled as equal to the one experienced by a flat plate of equal surface and Reynolds number. As a matter of fact the frictional drag on a flat plate in steady state laminar regime can be evaluated by means of the boundary layer theory yielding Blasius result $C_f = 1.328 \ R^{-1/2}$ which is experimentally shown to hold for $R \leq 3 \cdot 10^5$. In the turbulent regime, i.e. $R \geq 3 \cdot 10^5$ , the semiempirical equation of Schoenherr $0.242/\sqrt{C_f} = log_{10}(RC_f)$ holds. It is worthwhile noticing that even within this somehow ”ideal” framework of steady state flat plates there is a quite large domain of Reynold numbers, i.e. $R \in [10^5, 2 \cdot 10^6]$, in which the experimental data points reported in [34] are very scattered indicating that in that range of $R$ neither of the two models can be thought to be totally reliable. Notice that unfortunately many underwater robotic systems operate in that range of the Reynolds Number $R$. As far as the pressure drag coefficient $C_p$ is concerned, there is no general result of practical interest. It is usually assumed to be roughly independent from $R$ and it's value has to be determined experimentally for the particular body of interest.

Indeed the classical results outlined above are of little practical interest for underwater robotic applications. The experimental identification of the drag coefficients appears to be mandatory as even assuming to work with Reynold numbers far from the laminar/turbulent transition zone, which is highly unrealistic in the most common situations, the great majority of underwater robotic systems can not be modeled as slender bodies operating in steady state conditions.

#### 3.2.2.2 Lift forces

Lift forces are another consequence of viscosity. Generally speaking there are two kind of lift forces: *hydrofoil* and *vortex shedding* lift forces. A hydrofoil is a streamlined thin body that behaves as a lifting surfaces, i.e., that experience a force normal to its surface in a wing-like fashion. The lift force $F_{lift}$ applied to a hydrofoil of area $S$ in a fluid of density $\rho$ moving with steady state relative velocity $U$ can be modeled as $F_{lift} = \frac{1}{2}\rho U^2 S \ C_l(R, \alpha)$ being $\alpha$ the angle of attack, i.e., the angle between $U$ and the tangent to the surface $S$. As a rule of thumb the hydrofoil lift coefficient $C_l$ can be thought to be proportional to $\alpha$ for small values of $|\alpha|$, e.g. $|\alpha| < 10 \deg$, and sharply decaying to zero otherwise as for large values of the angle of attack stall occurs. The phenomenon of hydrofoil lift is of fundamental importance in a wide range of fluid dynamic applications as propellers, sails, wings, rudders and all kind of control surfaces. Nevertheless in all those situations where sharp surfaces are absent or the typical operating velocities are small, as for the majority of open frame bluff body ROVs or underwater manipulators, they can be reasonably neglected.



To qualitatively understand the phenomenon of vortex shedding consider a circular cylinder at rest in a still fluid. If the cylinder is suddenly accelerated to a constant regime speed normal to its axis separation of the flow will occur downstream. If upstream the flow may still be laminar, two initially symmetric vortices will start to grown in the downstream wake. These vortices can be shown to be unstable and in the final regime state of the cylinder they will be antisymmetric. The net result of the vortices instability is a periodic force normal to the cylinder axis and to its speed. This phenomenon is very important in many underwater systems: it is responsible for the strumming oscillations of cables and it may cause oscillations in many different kinds of underwater structures. As far as underwater robotic vehicles are concerned vortex shedding is usually neglected for slow motion open frame or bluff body vehicles. In principle fast slender body vehicles as many AUVs could be subject to vortex shedding periodic lift forces, but in practice it is not too difficult to employ small control surfaces in the downstream wake that limit the vortices correlation thus greatly reducing the overall vortex shedding lift effect. As far as underwater manipulators are concerned their cylindrical-like links could be reasonably subject to this phenomenon.

## 3.2.3 Added mass effects

The viscous effects described in paragraph (3.2.2) are not the only cause of forces applied to a rigid body moving in a fluid environment: indeed when a rigid body moves in an otherwise unbounded fluid it is expected to experience inertial forces related to the kinetic energy that the body itself induces on the whole bulk of fluid. These inertial forces have little to do with the viscosity properties of the fluid and for standard hydrodynamic Reynolds numbers ($R \geq 10^4$) they are described within the theory of ideal fluid.

### 3.2.3.1 On the properties of ideal fluids

A first important property of inviscid fluids is Lord Kelvin's theorem stating the "constancy of circulation in a circuit moving with the fluid in an inviscid fluid in which the density is either constant or is a function of the pressure" ( [35] pag. 84). Circulation $\Gamma$ on a closed circuit $c$ moving with the same fluid particles is defined as $\Gamma \triangleq \oint_c \mathbf{u}^T d\mathbf{x}$

being $\mathbf{u}$ the fluids velocity. The time derivative of $\Gamma$ is

$$\frac{d}{dt}\Gamma = \frac{d}{dt}\oint_c \mathbf{u}^T d\mathbf{x} = \oint_c (\frac{d}{dt}\mathbf{u}^T) d\mathbf{x} + \oint_c \mathbf{u}^T (\frac{d}{dt} d\mathbf{x}) \tag{3.27}$$

where the last integral is equal $\oint_c \mathbf{u}^T d\mathbf{u}$ which is zero. To evaluate the term $\oint_c (\frac{d}{dt}\mathbf{u}^T) d\mathbf{x}$ notice that the left hand side of the Navier Stokes equation (3.24) is exactly $\frac{d}{dt}\mathbf{u}^T$ . Neglecting viscosity, i.e. $\nu = 0$, and assuming that the only external force applied to the fluid is the conservative gravitational force, the right hand side of equation (3.24) can be written as $-\frac{1}{\rho}\nabla(p + \rho g h)$ being $g = 9.81 m/s^2$ the gravitational acceleration and $h$ the vertical Cartesian coordinate. Applying Stokes' theorem to equation (3.27) and replacing $\frac{d}{dt}\mathbf{u}^T = -\frac{1}{\rho}\nabla(p + \rho g h)$, the following holds:

$$\frac{d}{dt}\Gamma = \oint_c (\frac{d}{dt}\mathbf{u}^T) d\mathbf{x} = \int_S \left(\nabla \wedge \frac{d}{dt}\mathbf{u}\right)^T \mathbf{n}\, dS = -\int_S \nabla \wedge \left(\frac{1}{\rho}\nabla(p + \rho g h)\right) \cdot \mathbf{n}\, dS$$

being $S$ any surface bounded by the closed curve $c$. In the standard hypothesis of incompressible fluid (equation (3.21)) the last integral is equal to zero proving Kelvin's theorem $\frac{d}{dt}\Gamma = 0$. From a physical point of view Kelvin's theorem just states that in absence of shear dissipative stress and under the action of only conservative forces a circulation state of the fluid is a steady state. Assuming that the initial state of the circulation is $\Gamma = 0$ Kelvin's theorem implies that it remains null in time and by applying Stokes' theorem again

$$\oint_c \mathbf{u}^T d\mathbf{x} = \int_S (\nabla \wedge\ \mathbf{u})^T \mathbf{n}\, dS = 0\ \ \forall\, t \tag{3.28}$$

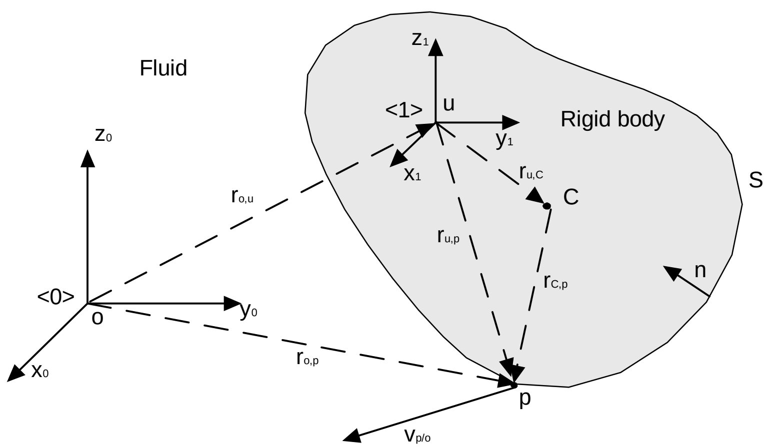
being $S$ any surface bounded by $c$. From the arbitrary of $S$ it follows that the integrand of the last integral must be identically null in time, thus showing that an inviscid incompressible fluid with no initial circulation is *irrotational*, i.e. $\nabla \wedge\ \mathbf{u} = 0$. Notice that equation (3.28) states that the velocity field $\mathbf{u}$ is conservative and can thus be written as the gradient of a scalar potential $\phi$, i.e. the velocity field of an inviscid incompressible fluid having initial circulation equal to zero can always be written as $\mathbf{u} = \nabla\phi$. Moreover notice that as in the same hypothesis equation (3.21) holds, the scalar $\phi$ must be a solution of Laplace equation $\nabla^2\phi = 0$, i.e. a *harmonic* function.

The Navier Stokes equation (3.24) for an inviscid fluid ($\nu = 0$) subject to the only gravitational force can now be written in terms of the velocity potential $\phi$ to yield Bernoulli's equation:

$$\begin{aligned} \frac{\partial}{\partial t}\nabla\phi + (\nabla\phi \cdot \nabla)\nabla\phi &= -\frac{1}{\rho}\nabla(p + \rho g h) \Rightarrow \\ \nabla(\frac{\partial}{\partial t}\phi + \frac{1}{2}\nabla\phi \cdot \nabla\phi) &= -\frac{1}{\rho}\nabla(p + \rho g h) \Rightarrow \end{aligned}$$

$$\frac{\partial}{\partial t}\phi + \frac{1}{2}\nabla\phi \cdot \nabla\phi + C(t) \;=\; -\frac{1}{\rho}(p + \rho g h) \tag{3.29}$$

being $C(t)$ constant with respect to the position, but possibly function of time. $C(t)$ is related to the additive constant of the velocity potential $\phi$ and can be always chosen to be zero. The two terms on the right hand side of equation (3.29) are sometimes referred to as the *dynamic pressure* $p$ and *hydrostatic pressure* $\rho g h$. The Bernoulli equation (3.29), as the Navier Stokes equation (3.24) for a viscid fluid, are general equations of motion that can describe a specific physical problem only if solved with the suitable boundary conditions. As a matter of fact all the information regarding the geometry of the problem is embedded in the boundary conditions. For a rigid body in a fluid environment the correct kinematic boundary condition to impose is that the fluid does not flow through the separating (moving) surface between itself and the body. With reference to figure




3.2.Rigid body underwater

(3.2), assuming $\mathbf{n}$ to be the unit vector normal to the separating surface pointing outside the fluid, i.e. inside the body, and $\mathbf{v}_{p/o}$ the velocity of a point $p$ on the separating surface defined as in equation (3.9), the above stated kinematic condition is satisfied if and only if the fluid velocity $\nabla\phi$ and the surface velocity $\mathbf{v}_{p/o}$ in every point $p$ have equal projection along $\mathbf{n}$, i.e.

$$\frac{\partial}{\partial \mathbf{n}}\phi = \mathbf{v}_{p/o}^T\mathbf{n} = (\mathbf{v}_{u/o} + \boldsymbol{\omega}_{1/0} \wedge \mathbf{r}_{u,p})^T\mathbf{n} \tag{3.30}$$

As it can be shown [34] that two different harmonic functions $\phi_1, \phi_2$ satisfying condition

(3.30) may differ only by a constant, equation (3.30) is actually just the right amount of ”extra” information needed to solve the Bernoulli equation (3.29). In a real viscous fluid, equation (3.30) should hold not for points $p$ on the separating surface of the rigid body, but for the points laying on the external face of the boundary layer within the fluid. Due to the negligeable thickness of the fluid boundary layer with respect to the rigid body for the great majority of robotic applications, it is reasonable to assume equation (3.30) to hold on the separating surface.

#### 3.2.3.2 Dynamic pressure forces and moments on a rigid body

Within the above developed theory of ideal or inviscid fluid the total force $\mathbf{F}_{dp}$ and moment $\mathbf{N}_{dp}$ experienced by a rigid body in a fluid media due to the only dynamic pressure can be written as

$$\mathbf{F}_{dp} = \int_S p\mathbf{n}\, dS = -\rho \int_S \left( \frac{\partial}{\partial t}\phi + \frac{1}{2}\nabla\phi \cdot \nabla\phi \right) \mathbf{n}\, dS \tag{3.31}$$

$$\mathbf{N}_{dp} = \int_S p(\mathbf{r}_{o,p} \wedge \mathbf{n})\, dS = -\rho \int_S \left( \frac{\partial}{\partial t}\phi + \frac{1}{2}\nabla\phi \cdot \nabla\phi \right) (\mathbf{r}_{o,p} \wedge \mathbf{n})\, dS \tag{3.32}$$

being $\phi$ a harmonic function subject to boundary conditions (3.30), $S$ the separating surface, $\mathbf{n}$ a unit normal vector and $\mathbf{r}_{o,p}$ a position vector as shown in figure (3.2). To explicitly solve equation (3.31) and (3.32) one more simplifying hypothesis is needed: the unboundedness of the fluid. As shown by Newman in [34] if the fluid is assumed to be unbounded except for the rigid body itself, equations (3.31) and (3.32) with boundary condition (3.30) can be solved analytically. To match the boundary condition (3.30) the total scalar velocity potential $\phi$ can be written in terms of a new vector $\boldsymbol{\psi} \in \Re^{6\times 1}$ and the generalized velocity $\boldsymbol{\nu}$ (defined by equation (3.15)) as

$$\phi = \boldsymbol{\psi}^T \boldsymbol{\nu} \tag{3.33}$$

The analytical solution of equations (3.31) and (3.32) as reported by Newman [34] yields for each component $j = 1, 2, 3$

$$F_{dp_j} = -\sum_{i=1}^{6} \left[ m_{ji} \frac{d_{<1>}}{dt} \nu_i + \sum_{k,l=1}^{3} \varepsilon_{jkl} \nu_i \omega_k m_{li} \right] \tag{3.34}$$

$$N_{dp_j} = -\sum_{i=1}^{6} \left[ m_{j+3,i} \frac{d_{<1>}}{dt} \nu_i + \sum_{k,l=1}^{3} \varepsilon_{jkl} \nu_i \omega_k m_{l+3,i} + \sum_{k,l=1}^{3} \varepsilon_{jkl} \nu_i \nu_k m_{li} \right] \tag{3.35}$$

being $\boldsymbol{\nu}$ the generalized velocity defined in equation (3.15), $\boldsymbol{\omega}$ the angular velocity $\boldsymbol{\omega}_{1/0}$, $\varepsilon_{jkl}$ the *Levi-Civita density* defined such that the $j^{th}$ component of the vector product between to given vectors $\mathbf{a}$ and $\mathbf{b}$ is $(\mathbf{a} \wedge \mathbf{b})_j = \sum_{k,l=1}^{3} \varepsilon_{jkl} a_k b_l$, i.e.

$$\begin{aligned} \varepsilon_{jkl} &= 1 \text{ if } jkl = \begin{cases} 1,2,3 \\ 2,3,1 \\ 3,1,2 \end{cases} \\ \varepsilon_{jkl} &= -1 \text{ if } jkl = \begin{cases} 1,3,2 \\ 2,1,3 \\ 3,2,1 \end{cases} \\ \varepsilon_{jkl} &= 0 \text{ otherwise} \end{aligned}$$

and $m_{ji}$ the components of the *added mass tensor* defined as

$$m_{ji} \triangleq \rho \int_S \psi_i \frac{\partial}{\partial \mathbf{n}} \psi_j \, dS \tag{3.36}$$

being $\psi_i$ the components of the $\boldsymbol{\psi}$ vector introduced in (3.33). Each component of $\boldsymbol{\psi}$ has to be harmonic (i.e., $\nabla^2 \psi_i = 0$) in the bulk of the fluid and has to satisfy the kinematic conditions

$$\frac{\partial \psi_i}{\partial n} = n_i \; \forall \; i = 1,2,3 \tag{3.37}$$

$$\frac{\partial \psi_i}{\partial n} = (\mathbf{r}_{u,p} \wedge \mathbf{n})_{i-3} \; \forall \; i = 4,5,6 \tag{3.38}$$

on the separating surface $S$. As a consequence each added mass component $m_{ji}$ given by equations (3.36) depends only on the shape of the boundary surface $S$ and on the constant (by hypothesis (3.21)) fluid density $\rho$. Equations (3.34) and (3.35) can be expressed in a more compact form writing the added mass tensor as

$$M_A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{3.39}$$

being each $M_{ij}$ a $3 \times 3$ matrix. With such notation equations (3.34) and (3.35) can be written as

$$\mathbf{F}_{dp} = -(M_{11} \; M_{12}) \frac{d_{<1>}}{dt} \boldsymbol{\nu} - \boldsymbol{\omega}_{1/0} \wedge [(M_{11} \; M_{12}) \, \boldsymbol{\nu}] \tag{3.40}$$

$$\begin{aligned} \mathbf{N}_{dp} = & -(M_{21} \; M_{22}) \frac{d_{<1>}}{dt} \boldsymbol{\nu} - \boldsymbol{\omega}_{1/0} \wedge [(M_{21} \; M_{22}) \, \boldsymbol{\nu}] \\ & - \mathbf{v}_{u/o} \wedge [(M_{11} \; M_{12}) \, \boldsymbol{\nu}] \end{aligned} \tag{3.41}$$

or in spatial notation $\boldsymbol{\tau}_{dp} \triangleq (\mathbf{F}_{dp}^T, \mathbf{N}_{dp}^T)^T$

$$\boldsymbol{\tau}_{dp} = -M_A \frac{d_{<1>}}{dt} \boldsymbol{\nu} - C_A(\boldsymbol{\nu}) \, \boldsymbol{\nu} \tag{3.42}$$

being

$$C_A(\boldsymbol{\nu}) \triangleq \begin{bmatrix} S(\boldsymbol{\omega}_{1/0}) & 0 \\ S(\mathbf{v}_{u/o}) & S(\boldsymbol{\omega}_{1/0}) \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{3.43}$$

Notice that from equations (3.40) and (3.41) follows that a rigid body subject to a constant linear velocity, i.e. $\mathbf{v}_{u/o} = const.$ and $\boldsymbol{\omega}_{1/0} = 0$, in an infinite inviscid fluid does not experience any dynamic pressure force (although it may experience a non zero moment due to the term $\mathbf{v}_{u/o} \wedge [(M_{11}\ M_{12})\ \boldsymbol{\nu}]$ in (3.41)); this fact is often referred to as the *D'Alembert paradox* in the fluid dynamic literature. Applying Green's theorem to the added mass components definition (3.36) it can be shown [34] that the added mass tensor $M_A$ of a rigid body in an ideal infinite fluid is symmetrical, i.e. $m_{ij} = m_{ji}$. Moreover starting from the energy conservation principle it can be shown [35] [34] that $M_A$ is related to the fluid kinetic energy by the quadratic form equation

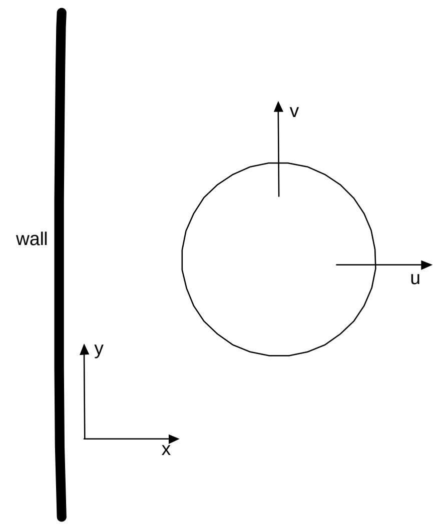
$$T_{fluid} = \frac{1}{2}\boldsymbol{\nu}^T\ M_A\ \boldsymbol{\nu}$$

being $\boldsymbol{\nu}$ the generalized rigid body velocity defined in equation (3.15). This property shows that $M_A$ is positive definite.

The practical limit of the above formulation describing the dynamic pressure on a whatsoever rigid body is related to the calculation of the added mass coefficients $m_{ij}$. These have been evaluated analytically (see for example [37]) only for very special geometries like spheroids or ellipsoids that are of very limited interest in real applications. To model real underwater system $M_A$ should be estimated experimentally. Moreover the above formulation has been derived with a number of ideal hypothesis that are here summarized:

(1) the body in the fluid is rigid
(2) the fluid is incompressible (equation (3.21))
(3) the fluid is ideal, i.e. inviscid, which implies Lord Kelvin's theorem and the irrotational nature of the fluid as derived in equation (3.28)
(4) the fluid is unbounded except for the rigid body itself

If the first two hypothesis are reasonably satisfied in most applications, the last two of them require some comments. The assumption of inviscid fluid for the derivation of the dynamic pressure forces is justified by the large value of the Reynolds number in the great majority of underwater robotic applications; of course viscous effects as drag and lift have to be taken into account by independent terms in the equation of motion of the system. As far as the last hypothesis is concerned an illustrative example of what happens to a sphere moving in the presence of an infinite fixed rigid wall is reported from [35]: if $x$ and $y$ are two Cartesian axes respectively normal and parallel to the wall (refer to figure (3.3)), a sphere of radius $r$ has a kinetic energy given to a first approximation by $T = \frac{1}{2}(Au^2 + Bv^2)$, being $u = \dot{x}$ and $v = \dot{y}$ the surge and sway velocities, $A = m + \frac{1}{2}m_f(1 + \frac{3}{8}(\frac{r}{x})^3)$, $B = m + \frac{1}{2}m_f(1 + \frac{3}{16}(\frac{r}{x})^3)$, $m$ the mass of the sphere and $m_f$ the mass of the displaced fluid. By applying Lagrange equations to the




3.3.Sphere inpresence of a wall

given kinetic energy the forces along the $x$ and $y$ axes follow

$$
\begin{aligned}
F_x &= \frac{9}{64} m_f \frac{r^3}{x^4}(-2u^2 + v^2) \\
F_y &= -\frac{9}{32} m_f \frac{r^3}{x^4}\, u\, v
\end{aligned}
$$

from which it is seen that the dynamic pressure force tends to repel the sphere if this moves at constant speed towards or away from the wall ($v = 0, u \neq 0$) and attracts it if the sphere tends to move parallel to the wall ($u = 0, v \neq 0$).

### 3.2.4 Current effects

Within the above described theory nothing have been said about eventual fluid currents. From the definition (3.33) and from the equations (3.37) and (3.38) it should be noticed that if the fluid is subject to a uniform motion $\mathbf{v}_{fluid/o}(t)$, the rigid body generalized velocity $\boldsymbol{\nu} = (\mathbf{v}_{u/o}^T, \boldsymbol{\omega}_{1/0}^T)^T$ appearing in equation (3.42) must be replaced by the relative velocity $((\mathbf{v}_{u/o} - \mathbf{v}_{fluid/0})^T, \boldsymbol{\omega}_{1/0}^T)^T$. As a matter of fact such a uniform current would also induce a buoyancy-like force, sometimes called *horizontal-buoyancy*, proportional to the product of the displaced fluid $m_f$ times the fluid acceleration $\frac{d_{<0>}}{dt}\mathbf{v}_{fluid/o}$. These forces are usually taken into account when underwater robotic systems are simulated [14] [38] [23] [39] but are usually neglected in control and identification schemes as it is very difficult to measure $\mathbf{v}_{fluid/o}$ and its time derivative. If the current is not uniform, as in presence of waves, the situation is even more complex as also the gradient of the fluid velocity is expected to produce a pressure force on the body. This latter phenomenon, which is fundamental in the modeling of surface systems, is generally neglected in deep water, but should be taken somehow into account in shallow water [40] [41] [42] [43].

### 3.2.5 Weight and buoyancy

Weight and buoyancy generalized force may be modeled as

$$\begin{pmatrix} \mathbf{F}_{wb} \\ \mathbf{N}_{wb} \end{pmatrix} = g \begin{bmatrix} -m\, I_{3\times3} & m_f\, I_{3\times3} \\ -m\, S(\mathbf{r}_{u,c}) & m_f\, S(\mathbf{r}_{u,B}) \end{bmatrix} \begin{pmatrix} {}^1\mathbf{k}_0 \\ {}^1\mathbf{k}_0 \end{pmatrix} \tag{3.44}$$

Being $m_f$ the displayed liquid volume, $m$ the rigid body mass, $g = 9.81\ m/s^2$ the gravitational acceleration, $\mathbf{r}_{u,B}$ the center of buoyancy local position vector, $\mathbf{r}_{u,c}$ the center of mass local position vector and ${}^1\mathbf{k}_0$ the projection of the $z$-axis inertial unit vector on the local reference $< 1 >$.

## 3.3 Underwater Remotely Operated Vehicles Model

The rigid body dynamic equations described in the previous sections can be viewed as the building blocks for more complex robotic system models as the ones of underwater vehicles or manipulators. In particular the dynamic models of a bluff body UUV will be derived. Generally bluff body UUVs are designed for low speed operations and are not equipped with lifting or control surfaces so their dynamic models do not take into account lift forces. The added mass and viscous drag effects are modeled on the basis of the rigid body theory described in the previous sections. Although drag is a distributed force on the surface of the vehicle for the sake of simplicity it is usually modeled within a lumped parameter formulation. The standard approach to drag modeling consists in the sum of a linear and quadratic term in the relative generalized six dimensional velocity $\boldsymbol{\nu}$, i.e.,

$$\mathbf{F}_{drag} = -D_{\boldsymbol{\nu}}\boldsymbol{\nu} - D_{\boldsymbol{\nu}|\boldsymbol{\nu}|}\boldsymbol{\nu}|\boldsymbol{\nu}| \tag{3.45}$$

being the matrixes $D_{\boldsymbol{\nu}}$ and $D_{\boldsymbol{\nu}|\boldsymbol{\nu}|}$ positive definite. A further and very common simplification [32] consists in assuming $D_{\boldsymbol{\nu}}$ and $D_{\boldsymbol{\nu}|\boldsymbol{\nu}|}$ diagonal thus neglecting the viscous drag coupling. The most common notation for the drag coefficients is

$$D_{\boldsymbol{\nu}} = diag(X_u, Y_v, Z_w, K_p, M_q, N_r) \tag{3.46}$$

$$D_{\boldsymbol{\nu}|\boldsymbol{\nu}|} = diag(X_{u|u|}, Y_{v|v|}, Z_{w|w|}, K_{p|p|}, M_{q|q|}, N_{r|r|}) \tag{3.47}$$

To obtain the complete model of a UUV thruster and cable dynamics are to be considered. The cable dynamics is sometimes modeled in simulation studies, but even being potentially a major source of drag or external applied force on an ROV, it is usually neglected in the design of ROV control systems. Indeed as for underwater currents, also cable forces are usually assumed to be disturbances of a nominal model that neglects them explicitly. From a practical point of view this can be an acceptable working hypothesis when the vehicle operates in a limited area, the cable is neutrally buoyant and is not in tension. If these conditions are not satisfied the cable forces applied on an ROV may be large and should be taken into account by an explicit term in the dynamic equation. As all the experimental data presented in this work has been collected matching the above stated working hypothesis regarding the cable, in the sequel its dynamics will not be taken explicitly into consideration but assumed to be a disturbance of the nominal model.

### 3.3.1 Thruster dynamics

As far as thruster dynamics is concerned a steady state equation can be obtained by dimensional analysis [34] yielding

$$\frac{T}{\rho n^2 d^4} = K(J) \tag{3.48}$$

being $J$ the advance ratio,

$$J = \frac{U}{nd} \begin{cases} U \text{ constant thruster velocity} \\ n \text{ number of revolutions per second} \\ d \text{ propeller diameter} \end{cases}$$

$T$ the thrust and $\rho$ the water density. In the great majority of the applications $K$ in equation (3.48) is assumed to be constant and the square dependance of $T$ on $n$ is written as $n|n|$ to take into account the sign of the thrust. Moreover in real application saturation occurs, thus the usual thrust model is assumed to be [32]

$$T = a\, n|n| - b\, n v_a \tag{3.49}$$

being $v_a$ the velocity of advance of the water through the propeller blades. The saturation term may be very important at high speeds, but is usually neglected in standard low speed operating conditions of ROVs. A dynamic thruster model taking into account the motor dynamics has been proposed by Yoerger et al. [44] and consists of the following equations:

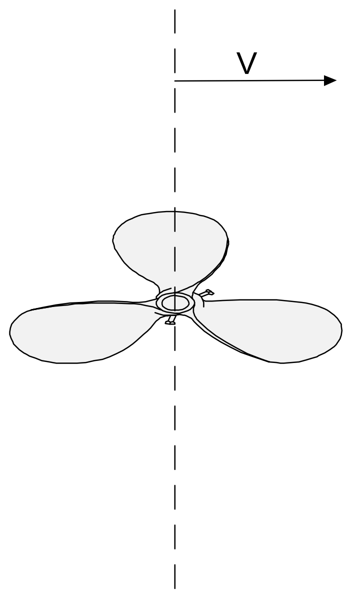
$$\begin{aligned} \frac{dn}{dt} &= \beta\, \tau - \alpha\, n|n| \\ T &= C_t\, n|n| \end{aligned}$$

being $\alpha$ and $\beta$ constants, and $\tau$ the input torque. Although the topic of thruster dynamic modeling and control has received a quite large attention in the past years [45]

[46] a simple steady state model as $T = a\, n|n|$ in which the propeller revolution rate is assumed to be linear in the applied DC motor voltage, thus neglecting the motor dynamics, is actually a very good approximation in all those applications in which $n$ does not suddenly change sign. In the sequel the thruster applied force will be modeled as

$$T = c\, V|V| \tag{3.50}$$

being $V$ the applied DC motor voltage and $c$ a constant to be experimentally identified. A difficulty related with this approach is that, as the identification of the $c$ constant in equation (3.50) is generally performed putting the single thruster in a cavitation tunnel and measuring the thrust as a function of the applied voltage, the propeller hull interactions are neglected. Indeed due to possible propeller hull interactions the operating conditions of the thruster in the cavitation tunnel may differ from the real ones especially, but not only, when open frame vehicles are considered. It will be shown in the next chapter by experimental data relative to the ROMEO open frame ROV that the propeller hull interaction may be significant and must be taken into account. Another kind of potentially important hydrodynamic ”interference” phenomenon regarding the thrusters dynamics is *momentum drag*. This phenomenon occurs when a thruster moves




3.4.Momentum drag

normally to its axis. With reference to figure (3.4) notice that in order to produce a flow parallel to the propeller axis, the fluid must be first accelerated to the same axis normal velocity $V$. This produces a drag force in the direction normal to the propeller axis that may be modeled as

$$\mathbf{F}_{md} = -\alpha n \mathbf{V} \tag{3.51}$$

being $n$ the propeller revolution rate, $\mathbf{V}$ the axis normal velocity and $\alpha$ a constant parameter. Momentum drag may be important in ROV systems as most of them are equipped with both horizontal and vertical thrusters that allow full translational control in $3D$ and that frequently operate together. Nevertheless to the knowledge of the author the literature relative to the modeling of such phenomenon in ROVs is limited to the only work of K. R. Goheen [38] and papers there cited.

### 3.3.2 Overall ROV Model

The complete model of an open frame UUV can be written as

$$\begin{aligned}(M + M_A)\frac{d_{<1>}}{dt}\boldsymbol{\nu} + [C(\boldsymbol{\omega}_{1/0}) + C_A(\boldsymbol{\nu})]\boldsymbol{\nu} + \\ +D_{\boldsymbol{\nu}}\boldsymbol{\nu} + D_{\boldsymbol{\nu}|\boldsymbol{\nu}|}\boldsymbol{\nu}|\boldsymbol{\nu}| - gW\,\mathbf{k} = \boldsymbol{\tau}_{th} + \boldsymbol{\delta}\end{aligned} \tag{3.52}$$

being all the terms defined as follows:

$$M \triangleq \begin{bmatrix} m\, I_{3\times3} & -m\, S(\mathbf{r}_{u,c}) \\ m\, S(\mathbf{r}_{u,c}) & I_u \end{bmatrix} \text{ as in equation (3.16)}$$

$$C(\boldsymbol{\omega}_{1/0}) \triangleq \begin{bmatrix} m\, S(\boldsymbol{\omega}_{1/0}) & -m\, S(\boldsymbol{\omega}_{1/0})\, S(\mathbf{r}_{u,c}) \\ m\, S(\mathbf{r}_{u,c})\, S(\boldsymbol{\omega}_{1/0}) & -S(I_u \boldsymbol{\omega}_{1/0}) \end{bmatrix} \text{ as in equation (3.17)}$$

$$M_A \triangleq \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \text{ as in equation (3.39)}$$

$$C_A(\boldsymbol{\nu}) \triangleq \begin{bmatrix} S(\boldsymbol{\omega}_{1/0}) & 0 \\ S(\mathbf{v}_{u/o}) & S(\boldsymbol{\omega}_{1/0}) \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \text{ as in equation (3.43)}$$

$$D_{\boldsymbol{\nu}} \triangleq diag(X_u, Y_v, Z_w, K_p, M_q, N_r) \text{ as in equation (3.46)}$$

$$D_{\boldsymbol{\nu}|\boldsymbol{\nu}|} \triangleq diag(X_{u|u|}, Y_{v|v|}, Z_{w|w|}, K_{p|p|}, M_{q|q|}, N_{r|r|}) \text{ as in equation (3.47)}$$

$$W \triangleq \begin{bmatrix} -m\, I_{3\times3} & m_f\, I_{3\times3} \\ -m\, S(\mathbf{r}_{u,c}) & m_f\, S(\mathbf{r}_{u,B}) \end{bmatrix} \text{ as in equation (3.44)}$$