Kinematic Time-invariant Control of a $2D$ Nonholonomic Vehicle

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Abstract

A closed loop, time-invariant and globally stable control law for a bicycle-like kinematic model is proposed. The resulting paths are smooth and the curvature bounded on the whole state space trajectories. As linear velocity can be kept arbitrary small, thus avoiding large lateral accelerations and actuator saturation problems, it is suggested that the proposed law may be also adopted for the planar control of autonomous underwater vehicles. The target configuration is always approached on a straight line and the vehicle is requested to move in only one specified forward direction thus avoiding cusps in the paths and satisfying a major requirement for the implementation of such strategy on many real systems.

1 Introduction

The main difficulty in the stabilization of a large class of nonholonomic systems is related to the theorem of Brockett [1] that shows how these systems cannot be stabilized by time-invariant smooth state feedback. A prototype of such systems is given by the cartesian unicycle kinematic model. As a consequence of both Brocketts result and of the interest of the topic among the mobile robotics research community the literature regarding the control of nonholonomic vehicles has widely grown and many control alternatives for either the kinematic or dynamic models of land, space or underwater nonholonomic robotic systems have been presented. Among these adaptive [2], neural net based [3], time-varying [4] [5] [6], and discontinuous [7] [8] [9] [10] [11] [12] [13] approaches have been proposed (for a wider description of the current state of the art in the control of nonholonomic systems refer to [14] [15] [16]). The discontinuous methodology is related to a remark of the above cited Brockett Theorem given by the same Brokcett [1]: *"If we have*

$$\dot{q} = \sum_{i=1}^{m} g_i(q)u_i : q(t) \in \Re^n$$

with the vectors $g_i(q)$ being linearly independent at q_0 then there exists a solution to the stabilization problem if and only if $m = n$. In this case we must have as many control parameters as we have dimensions of q. Of course the matter is completely different if the set $\{g_i(q_0)\}$ drops dimension precisely at q_0." As shown by the works of Casalino et al. [9], Badreddin et al.[7] and Astolfi [13], this last observation plays a key role in the solution of the unicycle stabilization problem: if the cartesian unicycle kinematics

$$\begin{aligned} \dot{x} &= u\cos\phi \\ \dot{y} &= u\sin\phi \\ \dot{\phi} &= \omega \end{aligned} \qquad (1)$$

being x and y the Cartesian coordinates of the robot, u its linear velocity in direction ϕ and ω the angular velocity is represented in polar-like coordinates (refer to figure(1))

$$\begin{aligned} e &\equiv \sqrt{x^2+y^2} \\ \theta &\equiv ATAN2(-y,-x) \\ \alpha &\equiv \theta-\phi \end{aligned} \qquad (2)$$

being $ATAN2(y,x) \in (-\pi,\pi]$ the four quadrant inverse tangent function, then the system state equation is

$$\dot{e} = -u\cos\alpha$$

$$
\begin{aligned}
\dot{\alpha} &= -\omega + u\frac{\sin\alpha}{e} \\
\dot{\theta} &= u\frac{\sin\alpha}{e}.
\end{aligned} \tag{3}
$$

As the state itself is not defined for $e = 0$, Brocketts Theorem does not hold anymore and a smooth time invariant state feedback law for global asymptotic stability is not prevented by Brocketts negative result. Indeed the idea of simply adopting a different (discontinuous) state representation in which Brocketts Theorem does not hold to solve the smooth state feedback global stability problem for general models of nonholonomic systems is very appealing and has been dealt by A. Astolfi [8] [12]. The problem of globally asymptotically stabilizing the system given by equation (3) has been solved by Casalino et al. and Aicardi et al. as accounted in [9] and [10] and similar results for the unicycle-like kinematic system have been presented by Astolfi [13]. There are two major draw backs of all these results that prevent their straightforward application to the control of the planar motion of real systems as underwater or air vehicles: (*i*) the unicycle-like nonholonomic constraint according to which the angular and linear velocities can be assigned independently and (*ii*) the fact that most real systems are allowed (or preferred) to move in only one given forward direction. As far as point (*i*) is concerned, this is equivalent to the obvious statement that a unicycle-like vehicle can turn on itself thus moving on an infinite curvature trajectory, while a wider class of moving systems (like bicycles, cars, autonomous underwater vehicles (AUVs) or airplanes) can only move on bounded curvature paths. In the present paper the results of Casalino et al. [9] are extended to the wider class of $2D$ nonholonomic vehicles moving in only one given forward direction and that cannot turn on the spot (*bicycle-like* kinematics).

2 Kinematic Model

With reference to figure (1), consider the cartesian kinematic model

$$
\begin{aligned}
\dot{x} &= u\cos\phi \\
\dot{y} &= u\sin\phi \\
\dot{\phi} &= u\frac{\tan\psi}{l} = uc
\end{aligned} \tag{4}
$$

describing the motion of the center of the rear wheel of a bicycle, being l the length of the robot, $\psi \in (-\pi/2, \pi/2)$ its steering angle, u the linear velocity and c the (bounded) curvature. Such simple model structure captures the main property of all those nonholonomic moving systems that cannot turn on them selves, i.e. being c bounded by hypothesis $\dot{\phi} = 0$ whenever $u = 0$ no matter the value of the steering input c. With the polar-like variable choice given in equation (2) this Cartesian model is transformed in

$$
\begin{aligned}
\dot{e} &= -u\cos\alpha \\
\dot{\alpha} &= -u\left(c - \frac{\sin\alpha}{e}\right) \\
\dot{\theta} &= u\frac{\sin\alpha}{e}
\end{aligned} \tag{5}
$$

on the whole state space except for the set $\Xi = \{e, \alpha, \theta : e = 0 \ \forall(\alpha, \theta) \in \Re^2\}$ where the model described by equation (5) is not defined. The problem consists in finding smooth time functions u and c that asymptotically drive the state (e, α, θ) towards the goal target $(0, 0, 0)$ on the boundary of the state space $\{e, \alpha, \theta\}$ from *any* initial configuration. Notice that with a suitable choice of the fixed reference frame with respect to which the state is defined the goal can always be thought of as the origin.

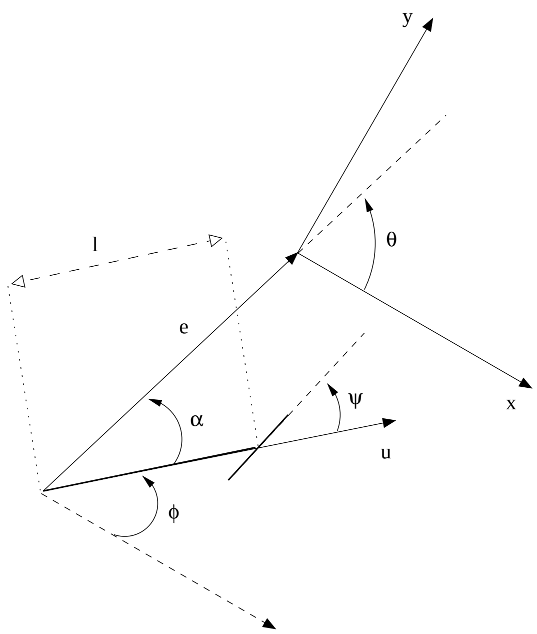

Figure 1: Bicycle-like kinematic model

3 Lyapunov-like based control law synthesis

As the state derivative (5) is identically null when $u = 0$ and in order to avoid sign changes in the linear velocity thus satisfying the requirement that the vehicle may move in only one given forward direction, suggests the law [17]:

$$
u = \gamma e \ : \ \gamma > 0. \tag{6}
$$

The point is to guarantee, by a suitable choice of c, that within some finite time $\cos\alpha > 0$ (so that e starts decreasing) and asymptotically $(e, \alpha, \theta) \to (0, 0, 0)$. To calculate c consider the state equation (5) given (6), i.e.,

$$
\begin{aligned}
\dot{e} &= -\gamma e \cos\alpha \\
\dot{\alpha} &= -\gamma e \left(c - \frac{\sin\alpha}{e} \right) \\
\dot{\theta} &= \gamma \sin\alpha
\end{aligned} \tag{7}
$$

and the quadratic Lyapunov candidate function

$$V \equiv \frac{1}{2}(\alpha^2 + h\theta^2) : h > 0 \tag{8}$$

having time derivative

$$\dot{V} = \alpha\dot{\alpha} + h\theta\dot{\theta} = \gamma(\alpha \sin\alpha + h\theta \sin\alpha - \alpha e c). \tag{9}$$

This last equation suggests the choice of c as:

$$c = \frac{\sin\alpha}{e} + h\frac{\theta}{e}\frac{\sin\alpha}{\alpha} + \beta\frac{\alpha}{e} : \beta > 0 \tag{10}$$

so that the time derivative of the candidate Lyapunov function V becomes

$$\dot{V} = -\gamma\beta\alpha^2 \leq 0. \tag{11}$$

Being V positive definite and radially unbounded equation (11) implies that V tends towards a non-negative finite limit, thus

$$
\begin{aligned}
\lim_{t\to\infty} \alpha &= \bar{\alpha} \\
\lim_{t\to\infty} \theta &= \bar{\theta}.
\end{aligned}
$$

The above and the fact that $\dot{V}$ is uniformly continuous ($\ddot{V} = -2\gamma\beta\alpha\dot{\alpha}$ is bounded) imply by Barbalat's Lemma that $\dot{V}$ tends to zero, so that $\bar{\alpha} = 0$. Substituting equation (10) in (7) gives:

$$
\begin{aligned}
\dot{e} &= -\gamma e \cos\alpha \\
\dot{\alpha} &= -\gamma \left(\beta\alpha + h\theta\frac{\sin\alpha}{\alpha} \right) \\
\dot{\theta} &= \gamma \sin\alpha.
\end{aligned} \tag{12}
$$

From the facts that $\alpha \to 0$, $\theta \to \bar{\theta}$, and that $\dot{\alpha}$ is uniformly continuous, again by Barbalat's Lemma it follows that the limit

$$\lim_{t\to\infty} \dot{\alpha} = -\gamma h \bar{\theta} = 0$$

and thus the limit value $\bar{\theta}$ of θ must be zero. Moreover notice from the last of equations (12) that given the above results also $\dot{\theta}$ tends asymptotically towards zero. The above results show that

$$
\begin{aligned}
\alpha &\to 0 \ ; \ \dot{\alpha} \to 0 \\
\theta &\to 0 \ ; \ \dot{\theta} \to 0
\end{aligned}
$$

so as $t \to \infty$ there must be some finite value of t, say t^*, starting from which $\cos\alpha > 0$ and thus e asymptotically exponentially converges to zero

$$\dot{e} \to -\gamma e \ \Rightarrow \ e \to 0.$$

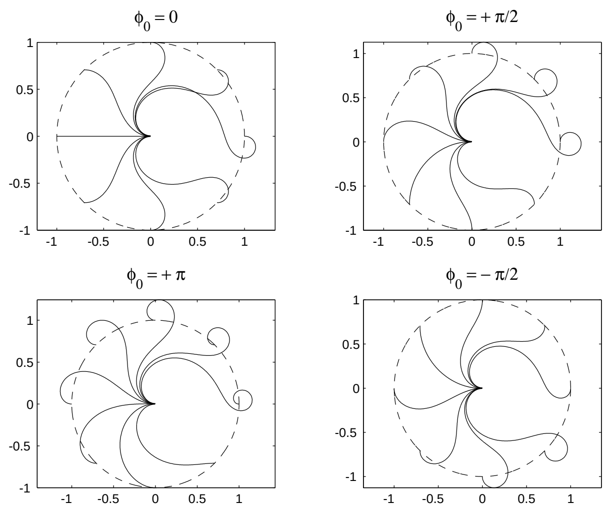

Figure 2: Paths starting on the unit circle with gains $\gamma = 1$, $h = 2$, $\beta = 2.9$ and unsaturated linear velocity. From the top left in clockwise direction the starting orientation ϕ_0 is: 0, $+\pi/2$, π, $-\pi/2$.

4 Some solution properties

The behaviour of the above developed closed loop control, i.e.,

$$
\begin{cases}
u = \gamma e : \gamma > 0 \\
c = \frac{\sin\alpha}{e} + h\frac{\theta}{e}\frac{\sin\alpha}{\alpha} + \beta\frac{\alpha}{e} : \beta, h > 0
\end{cases} \tag{13}
$$

depends on the choice of the parameters γ, β, h. In particular while u is obviously limited as long as e and γ are finite, the limit $\lim_{(e,\alpha,\theta)\to(0,0,0)} c$ must be analyzed in order to guarantee that c is bounded: when α approaches 0 the state equation (12) can be approximated by the linear system

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\theta} \end{pmatrix} = \begin{bmatrix} -\gamma\beta & -h\gamma \\ \gamma & 0 \end{bmatrix} \begin{pmatrix} \alpha \\ \theta \end{pmatrix} \tag{14}$$

$$\dot{e} = -\gamma e \tag{15}$$

and

$$c = \frac{\alpha}{e}(1+\beta) + h\frac{\theta}{e}$$

so that in order to reach the target $(0,0,0)$ on a straight line (i.e. with null curvature) the real part of the dominant pole of equation (14) must be strictly larger than γ. By direct calculation the eigenvalues of the system matrix of equation (14) are

$$\lambda_{\pm} = \frac{\gamma}{2}\left(-\beta \pm \sqrt{\beta^2 - 4h} \right) \tag{16}$$

and the requested condition $|Re(\lambda_+)| > \gamma$ is equivalent to

$$h > 1 \ ; \ 2 < \beta < h+1 \tag{17}$$

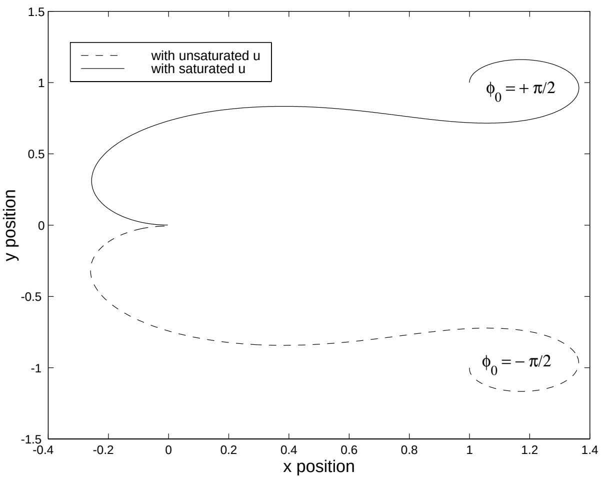

Figure 3: Paths with saturated (solid line, $\bar{u} = 1/4$) and unsaturated (dashed line) u for symmetrical initial configurations and same gains $\gamma = 1$, $\beta = 2.91$, $h = 2$. The variables α, θ, ϕ, u, c and e relative to this simulation are reported in figure (4).

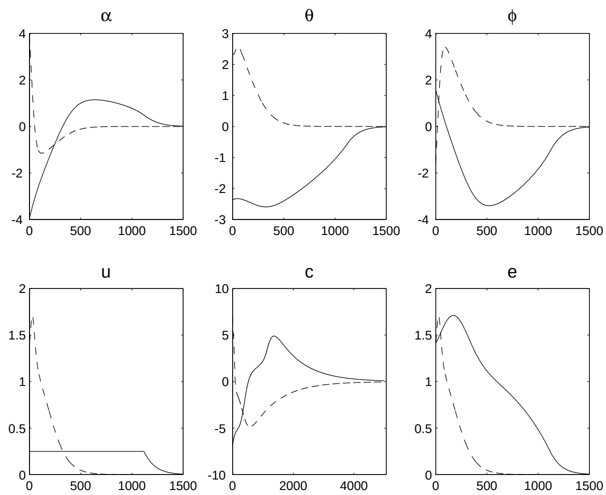

Figure 4: Saturated ($\bar{u} = 1/4$, solid lines) and unsaturated (dashed lines) results for the paths shown in figure (3).

thus, if h and β are chosen according to equation (17) c will be bounded on the whole state trajectory and will asymptotically converge to zero as requested.

A second interesting property of the proposed closed-loop solution regards the behaviour of α. According to equation (2) its initial value $\alpha|_{t=0}$ can take any real value as ϕ is unbounded. Nevertheless once that α reaches the set $[-n\pi, n\pi]$ from outside, being n any strictly positive integer, it will never leave it. This follows from the observation that according to equation (12)

$$\lim_{\alpha \to \pm n\pi} \dot{\alpha} = \mp \gamma \beta n \pi \tag{18}$$

5 Implementation issues

An important issue in the practical implementation of the control law (13) is related to the linear velocity u saturation that may occur if the target is too distant. Moreover in many applications, e.g. underwater vehicles, u should be kept lower than some upper bound $\bar{u}$ in order to avoid large lateral accelerations cu^2. Indeed global asymptotic convergence is also guaranteed replacing to $u = \gamma e$ in equation (13)

$$u = \gamma e \ \mathrm{sat}(\gamma e, \bar{u}) : \gamma > 0 \tag{19}$$

being

$$\mathrm{sat}(x, y) = \left\{ \begin{array}{l} 1 \ \forall \ x < y \\ \frac{y}{x} \ \forall \ x \geq y \end{array} \right. \quad \forall \ x, y > 0 \tag{20}$$

a positive and continuous saturation function that prevents the proportional control input u to grow larger than some prescribed upper bound $\bar{u}$. In such situation, i.e. applying the control law given by equations (19) and (10), the state equation is

$$\begin{aligned} \dot{e} &= -\gamma e \ \mathrm{sat}(\gamma e, \bar{u}) \cos \alpha \\ \dot{\alpha} &= -\gamma \ \mathrm{sat}(\gamma e, \bar{u}) \left(\beta \alpha + h \theta \frac{\sin \alpha}{\alpha} \right) \\ \dot{\theta} &= \gamma \ \mathrm{sat}(\gamma e, \bar{u}) \sin \alpha. \end{aligned} \tag{21}$$

Thus as long as the initial error $e|_{t=0}$ is finite and $\bar{u} > 0$ the convergence of the system (21) may be proved applying the Local Invariant Set Theorem [18] to the function

$$V_s \equiv e + \frac{1}{2}(\alpha^2 + h\theta^2) \tag{22}$$

$$\dot{V}_s = -\gamma \ \mathrm{sat}(\gamma e, \bar{u}) \left(e \cos \alpha + \beta \alpha^2 \right). \tag{23}$$

In particular defining $\Omega_l = \{(e, \alpha, \theta) : V_s < e_0\}$ then Ω_l is bounded and if

$$\beta \geq 4e_0 / 3\pi^2 \tag{24}$$

then $\dot{V}_s \leq 0$ in Ω_l. This last fact follows from the observation that $\cos \alpha + 4\alpha^2/3\pi^2 > 0 \ \forall \ \alpha$ and that e_0 is, by definition of Ω_l, an upper bound for e within Ω_l. As a consequence the state will converge to the largest invariant set M within the subset $R = \{(e, \alpha, \theta) \in \Omega_l : \dot{V}_s = 0\}$. By direct calculation it follows that with the above choice (24) for β, $R = \{(e, \alpha, \theta) : e = 0, \alpha = 0, \ \forall \ \theta\}$ and $M = \{(0, 0, 0)\}$ as if M was such that $\theta \neq 0, \alpha = 0$ then according to equation (21) $\dot{\alpha} \neq 0$ and M, by definition, wouldn't be an invariant set. This shows that if in addition to equation (17) required to reach the target on a straight line, β is chosen in accordance to equation (24), then the state will globally asymptotically converge to the target configuration no matter how small the maximum allowed linear speed $\bar{u}$ is. At last notice that all the properties described in Section 4 for the unsaturated case can be shown to hold also in the saturated case just replacing equations (21) to equations (12).

The above algorithms have been tested with some simulations that are reported in figures (2), (3), (4), (5) and (6).

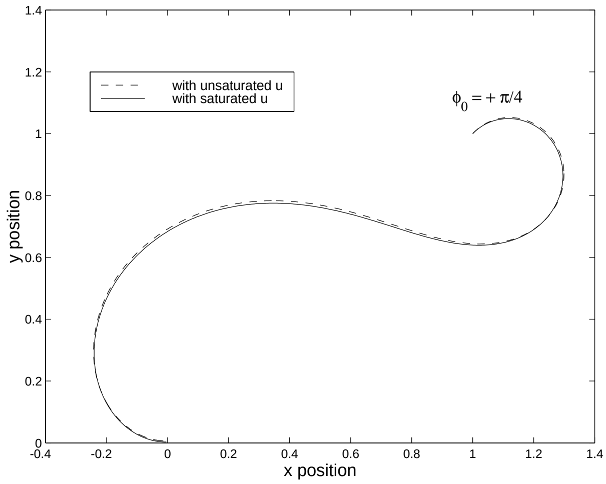

Figure 5: Paths with saturated (solid line, $\bar{u} = 1/2$) and unsaturated (dashed line) u for the same initial configuration $(1, 1, \pi/4)$ and same gains $\gamma = 1$, $\beta = 2.9$, $h = 2$.

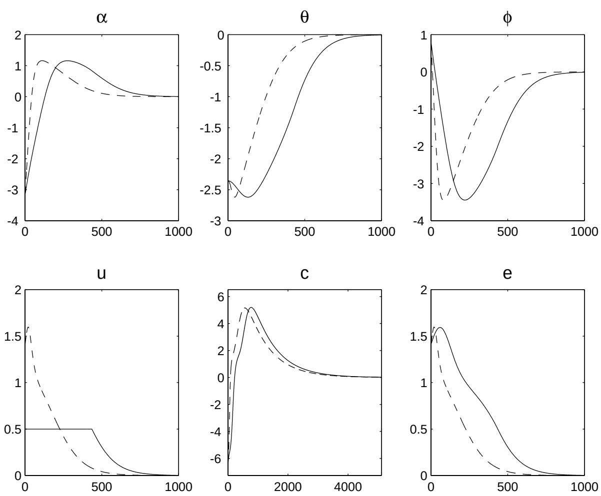

Figure 6: Saturated ($\bar{u} = 1/2$, solid lines) and unsaturated (dashed lines) results for the paths shown in figure (5).

6 Conclusions

A nonlinear, time-invariant, globally, exponentially and asymptotically converging control law has been presented for a general $2D$ nonholonomic kinematic model moving in only one given forward direction and taking into account steering constraints neglected in unicycle-like kinematic models. Thanks to the (discontinuous) polar-like mapping of the Cartesian model Brocketts Theorem does not prevent a time-invariant closed loop state feedback law generating *smooth* paths with a very limited computational burden with respect to alternative solution proposed in the literature. Moreover it has been shown that with a proper choice of the involved steering gains, the convergence properties of the proposed law are not affected by eventual linear velocity saturation. The curvature, the steering control input, is bounded on the whole state trajectory and lateral acceleration can be kept arbitrarily small at the expense of a slower convergence and higher steering gains. The curvature of the paths asymptotically converges to zero as the target configuration is approached and changes in the direction of the motion of the vehicle are never required thus avoiding cusps in the paths and satisfying a major requirement for the application of such control strategy to many real systems. It is suggested that such control law may result effective for the planar control of underwater vehicles.

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