# ON THE STABILIZATION OF THE UNICYCLE MODEL PROJECTING A HOLONOMIC SOLUTION

$8^{th}$ Int. Symposium on Robotics with Applications ISORA 2000, World Automation Congress WAC 2000, Maui, Hawaii, USA, June 2000

**M. Aicardi*, G. Cannata*, G. Casalino*, G. Indiveri†**

**DIST-Department of Communications Computers and Systems Science, Università di Genova, via all'Opera Pia 13, 16145 Genova, Italy*
{michele,cannata,pino}@dist.unige.it

† *GMD-AiS, German National Research Center on Information Technologies, Institute for Autonomous Intelligent Systems, Schloss Birlinghoven, 53754 Sankt Augustin, Germany*
giovanni.indiveri@gmd.de *(corresponding author)*

**ABSTRACT**

It is shown that, adopting a polar-like kinematic description of the Cartesian unicycle nonholonomic vehicle in order to prevent Brocketts negative result, a stabilizing time-invariant feedback law can be simply obtained projecting a suitable holonomic linear velocity on the nonholonomic linear velocity axis and superimposing an angular velocity that asymptotically steers such axis parallel to the holonomic velocity vector.

**KEYWORDS:** Nonholonomic vehicles, unicycle model

## INTRODUCTION

The problem of feedback stabilizing the nonholonomic unicycle-like system has received a wide attention in the literature of the last years. This was due both to the concern of the robotics community dealing with the practical control of nonholonomic vehicles and to the theoretical interest inspired by the work of Brockett [1] that shows how a large class of nonholonomic systems cannot be stabilized by time-invariant smooth state feedback. Nevertheless thanks to the results presented in [2] [3] [4] [5] it is by now clear that with a suitable choice of the state variables the determination of smooth time-invariant stabilizing feedback laws for the unicycle system is feasible. In particular Brocketts theorem requires the state equation to be continuous in the equilibrium point, thus if the state is described with a proper set of variables (i.e. such that the transformed state equation result *discontinuous* in the equilibrium point) Brocketts theorem does not hold any more and a smooth time-invariant feedback control law may well be obtained for the transformed "discontinuous" system. For a more detailed description of this *discontinuous* control approach for the asymptotic stabilization of nonholonomic systems refer to [6]. Given that a proper choice of the state variables allows a solution of the stabilization problem to be found, the

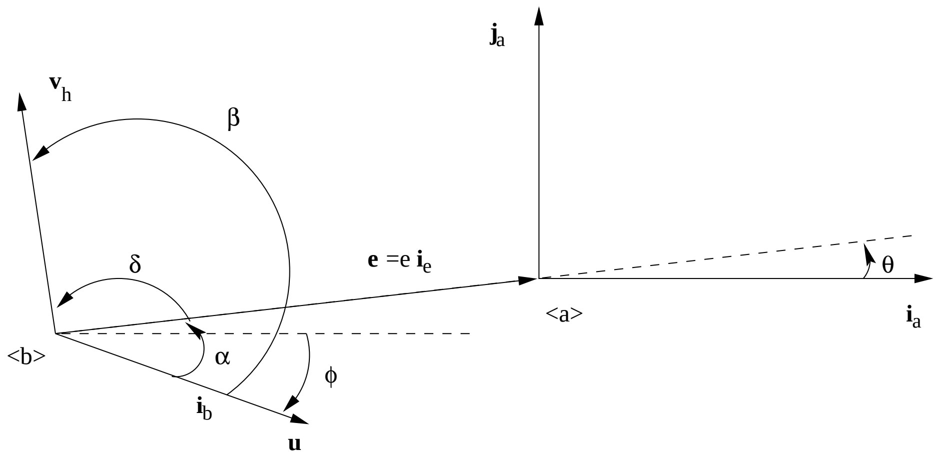


Figure 1: The model

question is posed on how such a control law can be synthesized. Previous works [2] [3] [4] show that standard nonlinear control methods based on Lyapunovs stability theory are indeed effective, but the resulting control strategies cannot be intuitively interpreted in relation to the equivalent stabilizing problem for a holonomic bidimensional system. In this paper it will be shown that asymptotic convergence of the unicycle systems can be guaranteed by a smooth time-invariant state feedback control law simply obtained projecting on the nonholonomic linear velocity axis a suitable "holonomic" velocity vector $\mathbf{v}_h$ and in the mean time applying an angular velocity that drives the nonholonomic linear velocity axis parallel to $\mathbf{v}_h$.

## CONTROL LAW SYNTHESIS

With reference to figure (1), consider the problem of driving an ideal single point system, such as the origin of the local reference frame $< b >$, to the origin of a given target frame $< a >$ without any nonholonomic constraint and such that the target frame is reached along its $x$ axis. This task may be accomplished applying, by example, the holonomic velocity

$$\mathbf{v}_h = \gamma_e \, e \, \mathbf{i}_e + \gamma_\theta \, \theta \, e \, \mathbf{j}_e \tag{1}$$

being $e = \|\mathbf{e}\|$, $\theta$ the angle between the $x$ axis of references $<a>$ and $\mathbf{e}$, $(\mathbf{i}_e, \mathbf{j}_e)$ the unit vectors of a reference frame having $x$ axis directed along $\mathbf{e}$s direction, $\mathbf{k}_e = \mathbf{k}_a$ and $\gamma_e > 0, \gamma_\theta > 0$ constant gains. Notice that by construction the velocity vector $\mathbf{v}_h$ given by equation (1) is the result of the superposition of an angular velocity $\dot{\theta} \, \mathbf{k}_e = -\gamma_\theta \, \theta \, \mathbf{k}_e$ and a linear velocity $\gamma_e \, e \, \mathbf{i}_e$ that guarantees exponential convergence of both $e$ and $\theta$ towards zero. In general such velocity vector $\mathbf{v}_h$ does not satisfy the unicycle nonholonomic constraint $\mathbf{u} \parallel \mathbf{i}_b$ expressed by the kinematic model:

$$\begin{aligned} \dot{x} &= u \cos \phi \\ \dot{y} &= u \sin \phi \\ \dot{\phi} &= \omega \end{aligned} \tag{2}$$

being $\phi \equiv \theta - \alpha$, $(x, y)$ the cartesian coordinates of the vehicle in the absolute reference $< a >$, $\omega$ and $u$ the norms of the angular and linear velocity vectors. The underlying guiding idea for the synthesis of a stabilizing time-invariant control law for the nonholonomic vehicle is to project at each time instant the constraint free velocity $\mathbf{v}_h$ along the direction of $\mathbf{u}$ and simultaneously compute an angular velocity $\omega\,\mathbf{k}_b$ that drives $\mathbf{u}$ parallel to $\mathbf{v}_h$. In accordance with this rationale, denoting by $(\mathbf{i}_b, \mathbf{j}_b)$ the local reference frame, the linear velocity $\mathbf{u}$ is computed as

$$\mathbf{u} = \mathbf{i}_b\, v_h \cos\beta = \mathbf{i}_b \sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}\;\; e \cos\beta, \tag{3}$$

being $v_h \equiv \|\mathbf{v}_h\|$ and $\beta$ the angle between the nonholonomic systems velocity vector $\mathbf{u}$ and the 'holonomic velocity' $\mathbf{v}_h$. In order to compute a suitable law for the angular velocity $\omega\,\mathbf{k}_b$ the time derivative of the angle $\beta$ needs to be analyzed. With reference to figure (1) the following holds:

$$\dot\beta = \dot\delta + \dot\alpha = \dot\delta + \dot\theta - \dot\phi \tag{4}$$

$$\dot\phi = \omega \tag{5}$$

$$\dot\theta = \frac{u \sin\alpha}{e} = \sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}\;\; \cos\beta\, \sin\alpha. \tag{6}$$

As far as $\dot\delta$ is concerned, consider

$${}^e\mathbf{e} \cdot {}^e\mathbf{v}_h \equiv \|\mathbf{v}_h\| \|\mathbf{e}\| \cos\delta = e^2 \sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}\;\; \cos\delta = \gamma_e\, e^2 \tag{7}$$

$${}^e\mathbf{e} \wedge {}^e\mathbf{v}_h \equiv {}^e\mathbf{k}_e \|\mathbf{v}_h\| \|\mathbf{e}\| \sin\delta = {}^e\mathbf{k}_e\, e^2 \sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}\;\; \sin\delta = {}^e\mathbf{k}_e\, \gamma_\theta\, \theta e^2, \tag{8}$$

being ${}^e\mathbf{v}_h$ given by equation (1) and ${}^e\mathbf{e} \equiv e\, \mathbf{i}_e$. As the unit vector $\mathbf{k}_e$ of the local reference is always well defined and parallel to the absolute unit vector $\mathbf{k}_a$, equations (7) and (8) imply that for every non null value of $e$ the following must hold:

$$\cos\delta = \frac{\gamma_e}{\sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}} \tag{9}$$

$$\sin\delta = \frac{\gamma_\theta\, \theta}{\sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}}. \tag{10}$$

Notice that $\delta$ depends only on $\theta$ and most important that $\delta \in (-\pi/2, \pi/2)\;\; \forall\, \theta \in (-\infty, \infty)$. Differentiating equation (9) and using equation (10) the time derivative of $\delta$ is found to be

$$\dot\delta = \frac{\gamma_e\, \gamma_\theta}{\gamma_e^2 + \gamma_\theta^2 \theta^2}\, \dot\theta \tag{11}$$

Given equations (4), (5) (6) and (11) the time derivative of $\beta$ is

$$\dot\beta = -\omega + \left(1 + \frac{\gamma_e\, \gamma_\theta}{\gamma_e^2 + \gamma_\theta^2 \theta^2}\right) \sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}\;\; \cos\beta\, \sin\alpha. \tag{12}$$

Given this result, $\omega$ may be chosen as

$$\omega = \gamma_\beta \beta + \left(1 + \frac{\gamma_e\, \gamma_\theta}{\gamma_e^2 + \gamma_\theta^2 \theta^2}\right) \sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}\;\; \cos\beta\, \sin\alpha \;\; : \;\; \gamma_\beta > 0 \tag{13}$$

that guarantees exponential convergence of $\beta$ to zero as the closed loop equation for $\beta$ is $\dot{\beta} = -\gamma_\beta \beta$. In order to reach the target along the desired direction, i.e. along $\mathbf{i}_a$, also $\theta, \phi, \alpha, \delta$ and $e$ must converge to zero. The time derivative of $\alpha = \theta - \phi$ can be written as

$$\dot{\alpha} = \dot{\theta} - \omega = -\gamma_\beta \, \beta - \frac{\gamma_e \, \gamma_\theta}{\sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}} \quad \cos\beta \, \sin\alpha \tag{14}$$

and thus the time derivative of $V_\alpha \equiv \frac{1}{2}\alpha^2$ turns out to be

$$\dot{V}_\alpha = -\gamma_\beta \, \alpha \, \beta - \frac{\alpha \, \gamma_e \, \gamma_\theta}{\sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}} \quad \cos\beta \, \sin\alpha \tag{15}$$

which, in general, is not semi-negative definite. Nevertheless $\beta$ tends asymptotically to zero and in the limit $\beta \to 0$ the following holds

$$\lim_{\beta \to 0} \dot{V}_\alpha = -\frac{\delta \, \gamma_e \, \gamma_\theta}{\sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)}} \quad \sin\delta \tag{16}$$

as $\lim_{\beta \to 0} \alpha = -\delta$. As it has been noticed previously if $\theta$ is finite $\delta$ can only take values in the *open* set $(-\pi/2, \pi/2)$, i.e. $\delta \in (-\pi/2, \pi/2) \;\; \forall \, \theta \in (-\infty, \infty)$ and in the set $(-\pi/2, \pi/2)$ the limiting value of $\dot{V}_\alpha$ given by equation (16) is negative definite. As a consequence of the above analysis it follows that the limit value of $\alpha, \delta$ and $\beta$ is zero and thus that $\phi \to \theta$. Indeed the convergence $\beta$ to zero implies that asymptotically the nonholonomic velocity $\mathbf{u}$ will converge to the velocity $\mathbf{v}_h$ and thus, by the very definition of $\mathbf{v}_h$ it follows that both $e$ and $\theta$ will converge to zero. In particular the asymptotic convergence of $e$ is also confirmed by the derivative of $V_e \equiv \frac{1}{2}e^2$, namely

$$\dot{V}_e = e\dot{e} = -eu\cos\alpha = -e^2\sqrt{(\gamma_e^2 + \gamma_\theta^2 \theta^2)} \quad \cos\beta \, \cos\alpha, \tag{17}$$

that tends to a negative limit as $\beta$ and $\alpha$ tend towards zero.

Notice that the angles $\alpha$, $\delta$ and $\theta$ are defined only when $e \neq 0$, thus the domain of attraction of the proposed law is $\mathcal{D} = \{(x, y, \phi) : (x, y) \neq (0, 0)\}$. Moreover if the initial state is in $\mathcal{D}$ then by replacing equation (3) in the state equation of $e$, i.e. $\dot{e} = -u\cos\alpha$, it follows that the proposed strategy keeps the state in $\mathcal{D}$ at all finite times. Convergence is not, strictly speaking, global, but it can be observed that the set $R^3 \setminus \mathcal{D}$ from which the proposed strategy fails to steer the system (2) to the origin has null measure. The presence of this set is an absolutely negligible fact for virtually all practical purposes and the proposed solution may be indeed considered "almost" global.

A most interesting property of the proposed strategy is that by simply choosing the gains $\gamma_e$, $\gamma_\theta$ and $\gamma_\beta$ such that $\gamma_e < \gamma_\theta < \gamma_\beta$ the curvature of the path will asymptotically tend to zero. This follows from the fact that once that $\beta$ has converged to zero the vehicle will actually follow the "holonomic" velocity $\mathbf{v}_h$ given by equation (1). The path described by such velocity will have null asymptotic curvature if $\theta$ converges faster than $e$. As in the limit $t \to \infty$ both $e$ and $\theta$ converge exponentially with time constants $1/\gamma_e$ and $1/\gamma_\theta$ it is sufficient that $\gamma_e < \gamma_\theta$ for the path described by $\mathbf{v}_h$ to have asymptotic null curvature. The condition $\gamma_\theta < \gamma_\beta$ guarantees that the exponential convergence of $\beta$ to zero is faster than the one of $\theta$.

## IMPLEMENTATION ISSUES AND SIMULATION EXAMPLES

The overall behaviour of the proposed algorithm has been analyzed by some simulations here reported for reference. The results shown in figure (2a) refer to the application of the control laws (3) and (13) with gains $\gamma_\theta = 0.3$, $\gamma_e = 0.1$ and $\gamma_\beta = 1$. In the top left plot the paths resulting from starting points on the unit circle with initial orientation $\phi_0 = \pi/2$ are visible. The time history of the controls $u$ and $\omega$ and of the variable $\beta$ for the path plotted with the thickest line are shown in the other plots of the figure (2a). The cusps in such paths are due to the $\cos\beta$ term in equation (3) that implies that the vehicle can move in both the forward and backwards directions and that $u$ can take the null value when $\omega$ does not (e.g. according to equations (3) and (13) $\beta = \pm\pi/2$ implies $u = 0$ and $\omega = \pm\gamma_\beta\pi/2$). Indeed some nonholonomic vehicles having a unicycle-like kinematic as bicycles, cars, surface vessels or underwater vehicles can not "turn on the spot" and/or move in two directions. For such kind of systems the proposed strategy can still be adopted noticing that the convergence analysis reported in the previous paragraph still holds when the $\cos\beta$ term in equations (3, 6, 12, 13, 14, 15, 17) is replaced by *any* continuous, strictly positive function $f(\beta)$ such that $f(0) = 1$. With such modification the linear velocity $u$ will be non null and of constant sign in the whole domain $\mathcal{D}$, the direction of motion is never reversed and cusps will be eliminated from the paths. Moreover as $u$ is non null and $\omega$ bounded in $\mathcal{D}$ the curvature $\omega/u$ is bounded in $\mathcal{D}$ and will tend to zero as $e \to 0$ as long as $\gamma_e < \gamma_\theta < \gamma_\beta$ as discussed above. Examples are visible in figure (2b) where $f = 1\ \forall\ \beta$, $\gamma_\theta = 0.3$, $\gamma_e = 0.1$ and $\gamma_\beta = 1$.

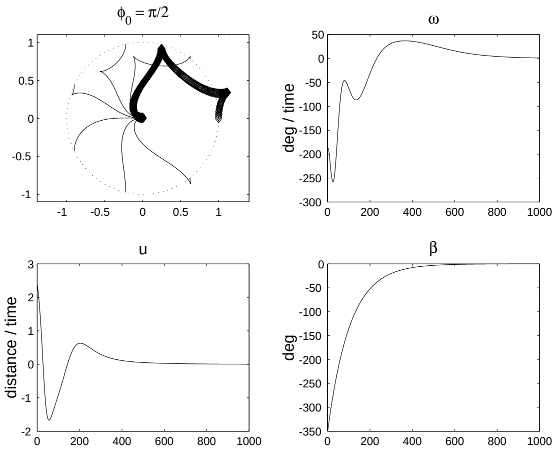


Figure 2a: Refer to text for details.

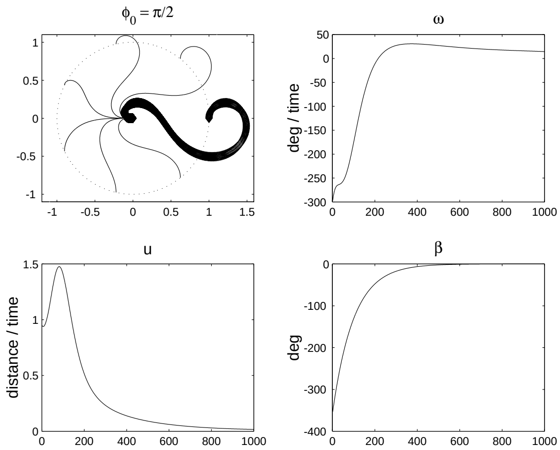


Figure 2b: Refer to text for details.

## CONCLUSIONS

A simple control strategy that guarantees asymptotic stability of the unicycle system has been presented. The algorithm takes advantage of the well known results of the discontinuous control approach [3] [6] for nonholonomic systems as the kinematic is described in polar-like coordinates [2] [4] [5]. The novel and perhaps interesting aspect of the proposed result relies in the use of a 'holonomic' velocity as a sort of reference signal for the stabilization of the nonholonomic system. Indeed the idea of projecting a holonomic solution to solve the motion planning problem for nonholonomic systems is not new [7]. Yet these past results were concerned with a planning issue rather than the closed-loop stabilization one and the projection operation was performed in a least squares sense rather than geometrically. The closed loop control law synthesis is performed in two steps. First an exponentially convergent constraint free velocity is computed for an ideal point systems, then the unicycle velocity is taken to be, at each instant, the projection of the constraint free ("holonomic") reference velocity on the systems velocity axis and in the mean time the unicycle's angular velocity is computed so that the systems axis steers exponentially parallel to the given 'holonomic' velocity. The derived control law has been proven to be asymptotically convergent by means of a Lyapunov-like analysis and simulations are reported to show the qualitative behaviour of the algorithm. Within this framework several additional objectives as moving in only one forward direction, avoiding cusps in the path, keeping the curvature bounded and asymptotically null can be satisfied.

## REFERENCES


1. Brockett, Millmann, and Sussmann, eds., *Differential Geometric Control Theory*, ch. Asymptotic Stability and Feedback Stabilization, by Brockett, R. W., pp. 181–191. Birkhauser, Boston, USA, 1983.
2. G. Casalino, M. Aicardi, A. Bicchi, and A. Balestrino, "Closed-loop steering for unicycle-like vehicles: A simple lyapunov like approach," in *IFAC Symposium on Robot Control, Sy.Ro.Co'94*, (Capri, Italy), pp. 291–298, September 1994.
3. A. Astolfi, "On the stabilization of non-holonomic systems," in *$33^{rd}$ Conference on Decision and Control, CDC'94*, (Buena Vista, FL, USA), pp. 3481–3486, 14-16 December 1994.
4. M. Aicardi, G. Casalino, A. Bicchi, and A. Balestrino, "Closed loop steering of unicycle-like vehicles via lyapunov techniques," *IEEE Robotics and Automation Magazine*, pp. 27–35, March 1995.
5. G. Indiveri, "Kinematic time-invariant control of a $2D$ nonholonomic vehicle," in *38th IEEE Conference on Decision and Control CDC'99*, (Phoenix, USA), December 1999.
6. A. Astolfi, "Discontinuous control of a nonholonomic system," *Systems & Control Letters*, vol. 27, pp. 37–45, 1996.
7. A. De Luca and G. Oriolo, "Local incremental planning for nonholonomic mobile robots," in *IEEE Int. Conference on Robotics and Automation, ICRA'94*, (S.Diego, USA), pp. 104–110, 1994.